Pricing Bermudan options under local Lévy models with default

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ICASQF 2016

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General Framework

We consider a defaultable asset S whose risk-neutral dynamics are given by:

$$
\begin{aligned}
S_t &= \mathbb{1}_{\{t<\zeta\}} e^{X_t}, \\
dX_t &= \mu(t, X_t)dt + \sigma(t, X_t)dW_t + \int_{\mathbb{R}} d\tilde{N}_t(t, X_{t-}, dz)z, \\
d\tilde{N}_t(t, X_{t-}, dz) &= dN_t(t, X_{t-}, dz) - \nu(t, X_{t-}, dz)dt, \\
\zeta &= \inf\{t \geq 0 : \int_0^t \gamma(s, X_s)ds \geq \varepsilon\},
\end{aligned} \tag{1}
$$

where $\tilde{N}_t(t, x, dz)$ is a compensated random measure with state-dependent Lévy measure $\nu(t, x, dz)$ and $\varepsilon \sim \mathrm{Exp}(1)$ and is independent of X.

Bermudan put option

Consider M exercise moments $\{t_1, ..., t_M\}$ with payoff at exercise time t_m to be $\phi(t_m, x)$. The option value $v(t, x)$ is defined recursively as

$$v(t_M, x) = \mathbb{1}_{\{\zeta > t_M\}} \phi(t_M, x),$$

and

$$\begin{cases} c(t, x) = E\left[e^{\int_t^{t_m} (r + \gamma(s, X_s)) ds} v(t_m, X_{t_m}) | X_t = x \right], \ t \in [t_{m-1}, t_m[\\ v(t_{m-1}, x) = \mathbb{1}_{\{\zeta > t_{m-1}\}} \max\{\phi(t_{m-1}, x), c(t_{m-1}, x)\}, \ m \in \{2, \ldots, M\}, \end{cases}$$

followed by

$$v(0, x) = c(0, x).$$

Approximation for expected values

With the COS method we calculate expected values (integrals):

$$v(t,x) = \int_{\mathbb{R}} \phi(T,y)\Gamma(t,x;T,dy),$$
$$\approx \sum_{k=0}^{N-1}{}' \operatorname{Re}\left(e^{-ik\pi\frac{a}{b-a}}\hat{\Gamma}\left(t,x;T,\frac{k\pi}{b-a}\right)\right)V_k(T),$$
$$V_k(T) = \frac{2}{b-a}\int_a^b \cos\left(k\pi\frac{y-a}{b-a}\right)\phi(T,y)dy,$$

where $\hat{\Gamma}$ is the characteristic function.

COS method for the Bermudan put

Remember we have

$$c(t,x) = e^{-r(t_m-t)} \int_{\mathbb{R}} v(t_m,y)\Gamma(t,x;t_m,dy), \qquad t \in [t_{m-1}, t_m[.$$

Using the Fourier-cosine expansion we get:

$$\hat{c}(t,x) = e^{-r(t_m-t)} {\sum_{k=0}^{N-1}}' \operatorname{Re}\left(e^{-ik\pi\frac{a}{b-a}} \hat{\Gamma}\left(t,x;t_m,\frac{k\pi}{b-a}\right)\right) V_k(t_m),$$

$$V_k(t_m) = \frac{2}{b-a}\int_a^b \cos\left(k\pi\frac{y-a}{b-a}\right)\max\{\phi(t_m,y), c(t_m,y)\}dy,$$

with $\phi(t,x) = (K-e^x)^+$ and $V_k(t_m)$ computed recursively.

Adjoint expansion of the characteristic function

The option price can be represented in integral form as

$$u(t,x) = \int_{\mathbb{R}} \phi(y)\Gamma(t,x;T,dy), \tag{2}$$

which solves the Cauchy problem

$$\begin{cases} Lu(t,x) = 0, & t \in [0,T[,\ x \in \mathbb{R}, \\ u(T,x) = \phi(x), & x \in \mathbb{R}, \end{cases} \tag{3}$$

where L is the integro-differential operator

$$\begin{aligned} L = {} & \partial_t + r\partial_x + \gamma(t,x)(\partial_x - 1) \\ & + \frac{\sigma^2(t,x)}{2}(\partial_{xx} - \partial_x) - \int_{\mathbb{R}} \nu(t,x,dz)(e^z - 1 - z)\partial_x \\ & + \int_{\mathbb{R}} \nu(t,x,dz)(e^{z\partial_x} - 1 - z\partial_x). \end{aligned}$$

A Taylor expansion of the coefficients

Use an expansion of the space-dependent coefficients in the operator L around some point $\bar{x}$.
Consider for simplicity only a local-volatility. Define

$$a(t,x) := \frac{\sigma^2(t,x)}{2}, \quad a_k = \frac{\partial_x^k a(\bar{x})}{k!}$$

The nth-order approximation of L is

$$L_n = L_0 + \sum_{k=1}^{n} \Big((x-\bar{x})^k a_k(\partial_{xx} - \partial_x) \Big),$$
$$L_0 = \partial_t + r\partial_x + a_0(\partial_{xx} - \partial_x).$$

Notice that

$$L_h - L_{h-1} = (x-\bar{x})^h a_h(\partial_{xx} - \partial_x).$$

Cauchy problems of the expansion

The nth-order approximation of Γ is defined as

$$\Gamma^{(n)}(t,x;T,y) = \sum_{k=0}^{n} G^k(t,x;T,y),$$

with G^0 solving

$$\begin{cases} L_0 G^0(t,x;T,y) = 0, \\ G^0(T,\cdot;T,y) = \delta_y. \end{cases}$$

and G^k for $k \geq 1$ defined through

$$\begin{cases} L_0 G^k(t,x;T,y) = -\sum_{h=1}^{k} (L_h - L_{h-1}) G^{k-h}(t,x;T,y), \\ G^k(T,x;T,y) = 0. \end{cases}$$

for $t \in [0,T[,\ x \in \mathbb{R}$

Solving the Adjoint Cauchy problems in Fourier space

The nth-order approximation of the characteristic function $\hat{\Gamma}$ is defined to be

$$\hat{\Gamma}^{(n)}(t,x;T,\xi) = \sum_{k=0}^{n} \mathcal{F}\left(G^k(t,x;T,\cdot)\right)(\xi) := \sum_{k=0}^{n} \hat{G}^k(t,x;T,\xi), \ \xi \in \mathbb{R}.$$

Note that Fourier transform is taken with respect to (T, y), but L acts on (t, x). We will:

- Define the functions $G^0(t,x;\cdot,\cdot)$ and $G^k(t,x;\cdot,\cdot)$, $k \geq 1$ through the Cauchy problems with the adjoint operator $\tilde{L}_0^{(T,y)}$ and $\tilde{L}_h^{(T,y)} - \tilde{L}_{h-1}^{(T,y)}$.
- Solve the adjoint Cauchy problems in the Fourier space. This immediately gives $\hat{\Gamma}$.

Theorem (Dual formulation)

The function $G^0(t,x;\cdot,\cdot)$ is defined through the following dual Cauchy problem

$$\begin{cases} \tilde{L}_0^{(T,y)} G^0(t,x;T,y) = 0 & T > t,\ y \in \mathbb{R}, \\ G^0(T,x;T,\cdot) = \delta_x. \end{cases}$$

For any $k \geq 1$the function $G^k(t,x;\cdot,\cdot)$ is defined through

$$\begin{cases} \tilde{L}_0^{(T,y)} G^k(t,x;T,y) = -\sum_{h=1}^{k} \left(\tilde{L}_h^{(T,y)} - \tilde{L}_{h-1}^{(T,y)}\right) G^{k-h}(t,x;T,y) \\ G^k(T,x;T,y) = 0 \end{cases}$$

with $\tilde{L}_0^{(T,y)}$ and $\tilde{L}_h^{(T,y)} - \tilde{L}_{h-1}^{(T,y)}$ being the adjoint operators.

Solution in Fourier space

We have

$$\tilde{L}_0^{(T,y)} = -\partial_T - r\partial_y + a_0(\partial_{yy} + \partial_y).$$

Then

$$\mathcal{F}\left(\tilde{L}_0^{(T,\cdot)} G^k(t,x;T,\cdot)\right)(\xi) = \psi(\xi)\hat{G}^k(t,x;T,\xi) - \partial_T \hat{G}^k(t,x;T,\xi),$$

where

$$\psi(\xi) = i\xi r + a_0(-\xi^2 - i\xi).$$

Then the solution to the adjoint Cauchy problems is given by

$$\hat{G}^0(t,x;T,\xi) = e^{i\xi x + (T-t)\psi(\xi)},$$

$$\hat{G}^k(t,x;T,\xi) = -\int_t^T e^{\psi(\xi)(T-s)} \mathcal{F}\left(\sum_{h=1}^{k} \left(\tilde{L}_h^{(s,\cdot)} - \tilde{L}_{h-1}^{(s,\cdot)}\right) G^{k-h}(t,x;s,\cdot)\right)(\xi) ds.$$

The characteristic function

The approximation of order n of the characteristic function is of the form

$$\hat{\Gamma}^{(n)}(t, x; T, \xi) := e^{i\xi x} \sum_{h=0}^{n} (x - \bar{x})^h g_{n,h}(t, T, \xi),$$

where the coefficients $g_{n,h}$, with $0 \leq h \leq n$, depend only on t, T and ξ, but not on x.

Back to the Bermudan option valuation [1/2]

Remember we had to value the continuation value of the form:

$$\hat{c}(t,x) = e^{-r(t_{m+1}-t)} \sum_{k=0}^{N-1}{}' \operatorname{Re}\left(e^{-ik\pi\frac{a}{b-a}} \hat{\Gamma}\left(t,x;t_{m+1},\frac{k\pi}{b-a}\right)\right) V_k(t_{m+1}),$$

$$V_k(t_m) = \frac{2}{b-a}\int_a^b \cos\left(k\pi\frac{y-a}{b-a}\right)\max\{\phi(t_m,y),c(t_m,y)\}dy.$$

We can rewrite

$$V_k(t_m) = \frac{2}{b-a}\int_{x_m^*}^b \cos\left(k\pi\frac{y-a}{b-a}\right)c(t_m,y)dy + C_k,$$

with x_m^* being the early-exercise point such that $c(t_m,x_m^*) = \phi(t_m,x_m^*)$.

Back to the Bermudan option valuation [2/2]

Inserting $\hat{c}(t,x)$ into the formula for $V_k(t_m)$ we find in vectorized form:

$$\hat{\mathbf{V}}(t_m) = \sum_{h=0}^{n} e^{-r(t_{m+1}-t_m)} \mathrm{Re}\left(\mathcal{M}^h(x_m^*, b)\boldsymbol{u}^h\right) + \mathbf{C}, \tag{4}$$

with

$$M_{k,j}^h(x_m^*, b) = \frac{2}{b-a}\int_{x_m^*}^{b} e^{ij\pi\frac{x-a}{b-a}}(x-\bar{x})^h \cos\left(k\pi\frac{x-a}{b-a}\right)dx \tag{5}$$

The matrix-vector multiplication $\mathcal{M}(x_m^*, b)\boldsymbol{u}$ can be calculated using a fast Fourier transform.

A quick example

Consider a process under the CEV-Merton dynamics with local vol. and Gaussian jumps.

Table: Prices for a European and a Bermudan Put option ($T = 1$ and 10 exercise dates) in the CEV-Merton model for the 2nd-order approximation of the characteristic function, and a Monte Carlo method.

	European		Bermudan	
K	MC 95% c.i.	Value	MC 95% c.i.	Value
0.8	0.02526-0.02622	0.02581	0.02617-0.02711	0.02520
1	0.08225-0.08395	0.08250	0.08480-0.08640	0.08593
1.2	0.1965-0.1989	0.1977	0.2097-0.2115	0.2132
1.4	0.3560-0.3589	0.3574	0.3946-0.3957	0.3954
1.6	0.5341-0.5385	0.5364	0.5930-0.5941	0.5932

For Further Reading

- A. Borovykh, C.W. Oosterlee, A.Pascucci,*Pricing Bermudan options under local Lévy models with default*, submitted, 2016
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- F. Fang and C.W. Oosterlee, *A novel option pricing method based on Fourier-cosine series expansions*, SIAM J. Sci. Comput. 31,2008, pp. 826-848.
- M.Lorig, S. Pagliarani, A. Pascucci, *A family of density expansions for Levy-type processes*, The Annals of Applied Probability, 25, 2015, pp. 235-267.