American swaptions under linear rational framework

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Outline

- Introduction
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Introduction (1/3)

References

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Introduction (2/3)

In the recent paper by Filipović, Larsson and Trolle new class of linear-rational term structure models was introduced, where

- The state price density (SPD) is modelled such that bond prices become linear-rational functions of the factors.
- This class is highly tractable with several distinct advantages: i) ensures non- negative interest rates, ii) easily accommodates unspanned factors affecting volatility and risk premiums, and iii) admits semi-analytical solutions to swaptions.
- A parsimonious model specification has a very good fit to both interest rate swaps and swaptions and captures many features of term structure, volatility, and risk premium dynamics.

Introduction (3/3)

In this paper we study the American swaption under the simplified linear-rational term structure model.

- This framework enables us to simplify the pricing problem significantly and reduce it to optimal stopping problem for a diffusion process.
- The latter problem is reduced to a free-boundary problem which we tackle by the local time-space calculus of Peskir.
- We characterize the optimal stopping boundary as the unique solution to nonlinear integral equation.
- Using these boundaries we obtain the arbitrage-free price of the American swaption and the optimal exercise strategies in terms of swap rate for both fixed-rate payer and receiver.

Model (1/3)

- LR term structure model includes two components: a factor process X_t and a SPD ζ_t as a deterministic function of the current state X_t.
- Here we assume that X is one-dimensional and is given by

$$dX_t = \kappa(\theta - X_t)\, dt + \sigma\sqrt{X_t}\, dB_t \quad (X_0 > 0) \tag{1}$$

where B is a SBM and $\kappa, \theta, \sigma > 0$ are positive constants, and SPD

$$\zeta_t = e^{-\int_0^t \alpha(s)ds}(1+X_t) \tag{2}$$

where the function $\alpha : [0, \infty) \mapsto \boldsymbol{R}$ is a deterministic continuous function, chosen such that the model-implied zero-coupon bond prices exactly match the observed term structure at time $t = 0$

Model (2/3)

- Under this model the following conditional expectations can be calculated:

$$\mathsf{E}_t X_T = \theta + e^{-\kappa(T-t)}(X_t - \theta) \tag{3}$$

$$\mathsf{E}_t \zeta_T = e^{-\int_0^T \alpha(s)ds}(1 + \theta + e^{-\kappa(T-t)}(X_t - \theta)) \tag{4}$$

for $0 \leq t \leq T$.

- Sufficient condition for the arbitrage-free market is the existence of SPD: a positive adapted process ζ_t such that the price $\Pi(t, T)$ at time t of any cash flow C_T at time T is given by

$$\Pi(t, T) = \frac{1}{\zeta_t}\mathsf{E}_t[\zeta_T C_T]. \tag{5}$$

Model (3/3)

- The main feature of LR framework is that it provides tractable expressions for ZCB prices $P(t,T)$ with $C_T = 1$:

$$P(t,T) = \frac{\mathsf{E}_t \zeta_T}{\zeta_t} = e^{-\int_t^T \alpha(s)ds} \frac{1 + \theta + e^{-\kappa(T-t)}(X_t - \theta)}{1 + X_t} \tag{6}$$

which explains why this model was referred as the linear-rational.

- The short rate is obtained via the formula $r_t = -\partial_T \log P(t,T)_{|T=t}$

$$r_t = \alpha(t) - \frac{\kappa(\theta - X_t)}{1 + X_t}. \tag{7}$$

Swap

We have a plain vanilla interest rate swap with payment dates $0 < T_0 < T_1 < ... < T_n$ such that $T_i - T_{i-1} = \Delta$ for $i = 1, .., n$ is a constant, and a pre-determined annualized rate K. At each date T_i, $i = 1, .., n$, the fixed leg pays K and the floating leg pays LIBOR accrued over the preceding time period. From the perspective of the fixed-rate payer, the value of the swap at time $t \leq T_0$ is then given by

$$\Pi_t^{swap} = P(t, T_0) - P(t, T_n) - \Delta K \sum_{j=1}^{n} P(t, T_j). \qquad (8)$$

European swaption

- A payer swaption is an option to enter into an interest rate swap, paying the fixed leg at a pre-determined rate and receiving the floating leg. A European payer swaption expiring at T_0 on a swap has a payoff at expiry T_0

$$C_{T_0} = (\Pi_{T_0}^{swap})^+ = \left(1 - P(T_0, T_n) - \Delta K \sum_{j=1}^{n} P(T_0, T_j)\right)^+. \tag{9}$$

- Under the linear-rational framework the price of European payer swaption at time $t \leq T_0$ equals

$$V_t^E = \frac{1}{\zeta_t} \mathrm{E}_t[\zeta_{T_0} C_{T_0}] = \frac{1}{\zeta_t} \mathrm{E}_t[g(X_{T_0})^+] \tag{10}$$

where $g(x)$ is the explicit linear function of x.

American swaption: problem formulation (1/6)

- Now we define the *American payer swaption* as an option to enter at any time T between T_0 and T_n into an interest rate swap. The value of swap at time $T \in [T_0, T_n]$

$$\Pi_T^{swap} = \sum_{m=1}^{n} \Big(1 - P(T, T_n) - (T_m - T) K P(T, T_m) - \Delta K \sum_{j=m+1}^{n} P(T, T_j) \Big) 1_{T_{m-1} \leq T < T_m}. \tag{11}$$

- Then the price of the American swaption at time T_0 is the value function of the optimal stopping problem

$$V_{T_0}^{A} = \frac{1}{\zeta_{T_0}} \sup_{0 \leq \tau \leq T_n - T_0} \mathrm{E}_{T_0} \left[\zeta_{T_0+\tau} (\Pi_{T_0+\tau}^{swap})^{+} \right]. \tag{12}$$

American swaption: problem formulation (2/6)

- In this paper we exploit a Markovian approach so that

$$V^A(t,x) = \frac{1}{\zeta_t} \sup_{0 \le \tau \le T_n - t} \mathsf{E}_{t,x}\left[\zeta_{t+\tau}(\Pi^{swap}_{t+\tau})^+\right] \tag{13}$$

  for $(t,x) \in [T_0, T_n] \times (0,\infty)$ and where the expectation $\mathsf{E}_{t,x}$ is taken under condition that $X_t = x$.

- The price $V^A_{T_0}$ at T_0 then equals $V^A(T_0, X_{T_0})$.
- Moreover, once (13) is determined, one can compute the price V^A_t at time $t \in [0, T_0)$ as

$$V^A_t = \frac{1}{\zeta_t}\mathsf{E}_t\left[\zeta_{T_0} V^A(T_0, X_{T_0})\right] \tag{14}$$

  using the known distribution of X_{T_0}.

American swaption: problem formulation (3/6)

The payoff of the optimal stopping problem when $t+\tau \in [T_{m-1}, T_m)$

$$\zeta_{t+\tau}(\Pi^{swap}_{t+\tau})^+ = \Big[G^1_m(t+\tau)X_{t+\tau} + G^2_m(t+\tau)\Big]^+ \qquad (15)$$

where functions G^1_m and G^2_m are given on intervals $[T_{m-1}, T_m)$ by

$$\begin{aligned}
G^1_m(t) &= \mathrm{e}^{-\int_0^t \alpha(s)ds} - c_n \mathrm{e}^{-\kappa(T_n - t)} - c_m(T_m - t)K\mathrm{e}^{-\kappa(T_m - t)} \\
&\qquad - \Delta K \sum_{j=m+1}^{n} c_j \mathrm{e}^{-\kappa(T_j - t)} \\
G^2_m(t) &= \theta\left(\widehat{G}^1_m(t) - G^1_m(t)\right) + \widehat{G}^1_m(t) \\
\widehat{G}^1_m(t) &= \mathrm{e}^{-\int_0^t \alpha(s)ds} - c_n - c_m(T_m - t)K - \Delta K \sum_{j=m+1}^{n} c_j \\
c_i &= \mathrm{e}^{-\int_0^{T_i} \alpha(s)ds}.
\end{aligned}$$

American swaption: problem formulation (4/6)

Therefore we can formulate the following optimal stopping problem

$$V(t,x) = \sup_{0 \le \tau \le T_n - t} \mathsf{E}_x \left[G^+(t+\tau, X_\tau) \right] \tag{16}$$

for $(t,x) \in [T_0, T_n] \times (0, \infty)$ and function G is given by

$$G(t,x) = \sum_{m=1}^{n} \left(G_m^1(t)x + G_m^2(t) \right) 1_{T_{m-1} \le t < T_m} = G^1(t)x + G^2(t)$$

and we then get

$$V^A(t,x) = V(t,x)/\zeta_t = e^{\int_0^t \alpha(s)ds} V(t,x)/(1+X_t) \tag{17}$$

for $(t,x) \in [T_0, T_n] \times (0, \infty)$ so that we focus on the problem (17).

American swaption: problem formulation (5/6)

It is important to note that $G(T_n, x) = 0$ for all $x > 0$ and hence it is not optimal to enter into the swap when $G \leq 0$ as with positive probability we can enter later into the set where $G > 0$. This observation allows us to simplify (16) by removing the positive part and formulate the equivalent problem

$$V(t, x) = \sup_{0 \leq \tau \leq T_n - t} \mathsf{E}_x G(t+\tau, X_\tau) \tag{18}$$

for $(t, x) \in [T_0, T_n] \times (0, \infty)$.

Free-boundary problem (1/6)

In this section we will reduce the problem (18) into a free-boundary problem and the latter will be tackled using local time-space calculus of Peskir. Using that the gain function $G(t,x)$ is continuous and standard arguments we have that continuation and stopping sets read

$$C^* = \{ (t,x) \in [T_0, T_n) \times (0,\infty) : V(t,x) > G(t,x) \} \tag{19}$$

$$D^* = \{ (t,x) \in [T_0, T_n) \times (0,\infty) : V(t,x) = G(t,x) \} \tag{20}$$

and the optimal stopping time in (18) is given by

$$\tau^* = \inf \{ 0 \le s \le T_n - t : (t+s, X_s^x) \in D^* \}. \tag{21}$$

Free-boundary problem (2/6)

- **Assumption.** We assume that $\kappa + \underline{\alpha} > K$, where $\underline{\alpha} = \inf_{t>0} \alpha(t)$, which is empirically very natural and acceptable.

- **Properties of G^1 and G^2:**

  1. $G^1(T_n) = G^2(T_n) = 0$, $G^1(t) > G^2(t)$ for all $t \in [T_0, T_n)$.
  2. G^1 and G^2 are continuous on $[T_0, T_n)$, however their derivatives are discontinuous at payment dates T_m, $m = 1, ..., n-1$.
  3. The function G^1 is positive on $[T_0, T_n)$.
  4. $\lim_{t \uparrow T_n} \frac{G^1(t)}{T_n - t} = c_n (\alpha(T_n) + \kappa - K) > 0$
  5. $\lim_{t \uparrow T_n} \frac{G^2(t)}{T_n - t} = c_n (-\theta\kappa + \alpha(T_n) - K)$.

Free-boundary problem (3/6)

We calculate the function of instantaneous benefit of waiting to exercise $H(t,x) = (G_t + \boldsymbol{L}_X G)(t,x)$ for $(t,x) \in [T_0, T_n) \times (0,\infty)$ where $\boldsymbol{L}_X = \kappa(\theta - x) d/dx + (\sigma^2/2)\, x\, d^2/dx^2$ is the infinitesimal generator of X and we have that

$$H(t,x) = \sum_{m=1}^{n} \left(H_m^1(t)x + H_m^2(t)\right) 1_{T_{m-1} \le t < T_m}$$

for $t \in [T_{m-1}, T_m)$, $m = 1, \ldots, n$ and where

$$H_m^1(t) = -\left(\kappa + \alpha(t)\right) \mathrm{e}^{-\int_0^t \alpha(s)ds} + c_m K e^{-\kappa(T_m - t)} < 0 \qquad (22)$$

$$H_m^2(t) = -\theta H_m^1(t) + (1+\theta)\left(c_m K - \alpha(t)\mathrm{e}^{-\int_0^t \alpha(s)ds}\right). \qquad (23)$$

Free-boundary problem (4/6)

- Using Itô-Tanaka formula we get

$$\mathsf{E}G(t+\tau, X^x_\tau) = G(t,x) + \mathsf{E}\int_0^\tau H(t+s, X^x_s)ds \tag{24}$$

for $(t,x) \in [T_0, T_n) \times (0,\infty)$ where the integral term with respect to the local time is not present since the underlying process, time, is of bounded variation. It is obvious that (24) indicates that the set $\{(t,x) \in [T_0, T_n) \times (0,\infty) : H(t,x) > 0\}$ belongs to continuation set C^*.

- We prove that there exists a function $b : [T_0, T_n) \to (0,\infty)$ such that

$$D^* = \{\,(t,x) \in [T_0, T_n) \times (0,\infty) : x \geq b(t)\,\}. \tag{25}$$

Free-boundary problem (5/6)

- It can be seen that there exist curves γ and h on $[T_0, T_n]$ defined as

$$G(t, \gamma(t)) = 0 \quad \text{and} \quad H(t, h(t)) = 0 \tag{26}$$

for $t \in [T_0, T_n)$ (Figure 1) such that $G(t, x) > 0$ for $x > \gamma(t)$ and $G(t, x) < 0$ for $x < \gamma(t)$, $H(t, x) > 0$ for $x < h(t)$ and $H(t, x) < 0$ for $x > h(t)$. Since it is not optimal to stop when $G < 0$ and $H > 0$, we have $b > \gamma \vee 0$ and $b > h \vee 0$ on $[T_0, T_n)$ as X_t is always positive.

- Simple arguments show that $b(T_n-) = h(T_n-) \vee 0$ where

$$h(T_n-) = -\frac{H^2(T_n-)}{H^1(T_n-)} = \frac{\theta\kappa - \alpha(T_n) + K}{\alpha(T_n) + \kappa - K} \tag{27}$$

Free-boundary problem (6/7)

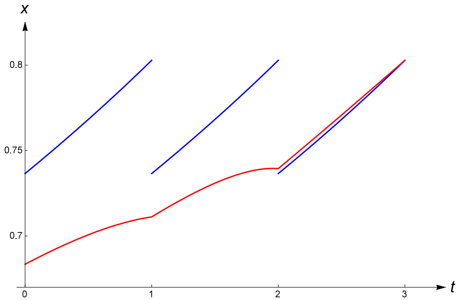

Figure 1. A computer drawing of function γ (red) and h (blue). The parameter set is $T_0 = 0, \Delta = 1, n = 3, \theta = 2.66, \kappa = 0.03, \sigma = 0.29, \alpha = \theta\kappa = 0.08, K = 0.05$.

Free-boundary problem (7/7)

Standard Markovian arguments lead to the following free-boundary problem for $V = V(t, x)$ and $b = b(t)$ to be determined:

$$V_t + \boldsymbol{L}_X V = 0 \qquad \text{in } C^* \tag{28}$$

$$V(t, b(t)) = G(t, b(t)) \qquad \text{for } t \in [T_0, T_n) \tag{29}$$

$$V_x(t, b(t)) = G_x(t, b(t)) \qquad \text{for } t \in [T_0, T_n) \tag{30}$$

$$V(t, x) > G(t, x) \qquad \text{in } C^* \tag{31}$$

$$V(t, x) = G(t, x) \qquad \text{in } D^* \tag{32}$$

where the continuation set C^* and the stopping set D^* are given by

$$C^* = \{\, (t, x) \in [T_0, T_n) \times (0, \infty) : x < b(t) \,\} \tag{33}$$

$$D^* = \{\, (t, x) \in [T_0, T_n) \times (0, \infty) : x \geq b(t) \,\}. \tag{34}$$

American payer swaption (1/3)

We now provide the early exercise premium representation formula for the value function V which decomposes it into the sum of the expected payoff with exercise at T_n (which is zero) and early exercise premium which depends on the boundary b. We define the following function

$$L(t,u,x,z) = -\,\mathsf{E}\left[H(t+u, X_u^x) I(X_u^x \geq z)\right] \tag{35}$$
$$= -\int_z^\infty H(t+u, \widehat{x}) q(\widehat{x}; u, x) d\widehat{x}$$

for $t, u \geq 0$ and $x, z > 0$ where $q(\widehat{x}; u, x)$ is a non-central chi-squared pdf.

American payer swaption (2/3)

Theorem. The value function V has the following representation

$$V(t,x) = \int_0^{T_n - t} L(t,u,x,b(t+u))du \tag{36}$$

for $t \in [T_0, T_n)$ and $x \in (0,\infty)$. The optimal stopping boundary b can be characterized as the unique solution to the nonlinear integral equation

$$G(t,b(t)) = \int_0^{T_n - t} L(t,u,b(t),b(t+u))du \tag{37}$$

for $t \in [T_0, T_n)$ in the class of continuous functions $t \mapsto b(t)$ with $b \geq h \vee 0$.

American payer swaption (3/3)

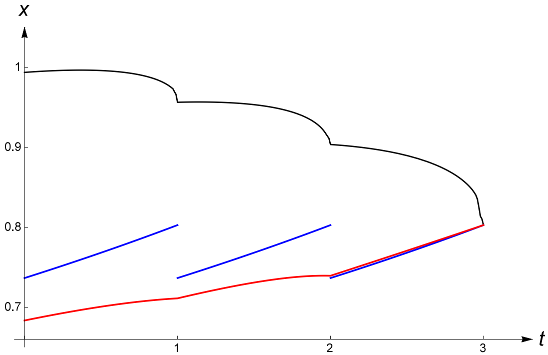

Figure 2. A computer drawing of the optimal stopping boundary $t \mapsto b(t)$ (black). The parameter set is $T_0 = 0$, $\Delta = 1$ year, $n = 3$, $\theta = 2.66$, $\kappa = 0.03$, $\sigma = 0.29$, $K = 0.05$, $\alpha \equiv \theta\kappa = 0.08$.

Proof (1/2)

Proof. By applying the local time-space formula on curves (Peskir, 2005) for $V(t+s, X_s^x)$ we have that

$$
\begin{aligned}
V(t+s,X_s^x) & \qquad (38)\\
&= V(t,x) + M_s \\
&\quad + \int_0^s (V_t + \mathbf{L}_X V)(t+u, X_u^x) I(X_u^x \neq b(t+u))\,du \\
&\quad + \frac{1}{2}\int_0^s \big(V_x(t+u, b(t+u)+) - V_x(t+u, b(t+u)-)\big)\, d\ell_u^b(X^x) \\
&= V(t,x) + M_s + \int_0^s (G_t + \mathbf{L}_X G)(t+u, X_u^x) I(X_u^x \geq b(t+u))\,du \\
&= V(t,x) + M_s + \int_0^s H(t+u, X_u^x) I(X_u^x \geq b(t+u))\,du
\end{aligned}
$$

Proof (2/2)

where we used PDE for V in C^*, the definition of H, the smooth-fit condition and where $M = (M_u)_{u \geq 0}$ is the martingale term, $(\ell_u^b(X^x))_{u \geq 0}$ is the local time process of X^x spending at the boundary b. Now upon letting $s = T_n - t$, taking the expectation E, using the optional sampling theorem for M, rearranging terms and noting that $V(T_n, x) = G(T_n, x) = 0$ for all $x > 0$, we get (36). The integral equation (37) is obtained by inserting $x = b(t)$ into (36) and using continuous fit property.

Numerical algorithm

- Set $t_k = kh$ for $k = 0, 1, \dots, N$ where $h = (T_n - T_0)/N$ so that we can approximate the integral equation above as

$$G(t_k, b(t_k)) = h \sum_{l=k+1}^{N} L\big(t_k, t_l - t_k, b(t_k), b(t_l)\big) \tag{39}$$

  for $k = 0, 1, \dots, N-1$. Setting $k = N-1$ and $b(t_N) = h(T_N) \vee 0$ we can solve this equation numerically and get number $b(t_{N-1})$. Continuing this, we obtain $b(t_N), b(t_{N-1}), \dots, b(t_1), b(t_0)$ as approximations of b at the points $T_n, T_n - h, \dots, T_0 + h, T_0$.

- The value function for $k = 0, 1, \dots, N-1$ and $x > 0$ is computed as

$$V(t_k, x) = h \sum_{l=k+1}^{N} L\big(t_k, t_l - t_k, x, b(t_l)\big). \tag{40}$$

American receiver swaption (1/5)

Now we turn to *American receiver swaption* which is an option to enter at any time T between T_0 and T_n into an interest rate swap, receiving the fixed leg at a pre-determined rate K and paying the floating leg. By doing similar manipulations as above we formulate

$$\widetilde{V}(t,x) = \inf_{0 \le \tau \le T_n - t} \mathrm{E}_x G(t+\tau, X_\tau) \tag{41}$$

and the price of American receiver swaption is

$$\widetilde{V}^A(t,x) = -\mathrm{e}^{\int_0^t \alpha(s)ds} \widetilde{V}(t,x)/(1+X_t) \tag{42}$$

for $(t,x) \in [T_0, T_n] \times (0,\infty)$.

American receiver swaption (2/5)

The continuation and stopping sets are now following

$$\widetilde{C}^* = \{\,(t,x) \in [T_0, T_n) \times (0,\infty) : \widetilde{V}(t,x) < G(t,x)\,\} \tag{43}$$

$$\widetilde{D}^* = \{\,(t,x) \in [T_0, T_n) \times (0,\infty) : \widetilde{V}(t,x) = G(t,x)\,\} \tag{44}$$

and we can show that there is a function $\widetilde{b} : [T_0, T_n) \to (0, \infty)$ such that

$$\widetilde{D}^* = \{\,(t,x) \in [T_0, T_n) \times (0,\infty) : x \le \widetilde{b}(t)\,\}. \tag{45}$$

Since the problem is the minimization one, we should not stop when $G > 0$ and $H < 0$, i.e. we have that $\widetilde{b} < \gamma$ and $\widetilde{b} < h$ on $[T_0, T_n)$. The terminal value of $\widetilde{b}$ is $\widetilde{b}(T_n-) = h(T_n-) \vee 0 = b(T_n-)$.

American receiver swaption (3/5)

Associated free-boundary problem for $\widetilde{V} = \widetilde{V}(t, x$ and $\widetilde{b} = \widetilde{b}(t)$:

$$\widetilde{V}_t + \boldsymbol{L}_X \widetilde{V} = 0 \qquad \text{in } \widetilde{C}^* \tag{46}$$

$$\widetilde{V}(t, \widetilde{b}(t)) = G(t, \widetilde{b}(t)) \qquad \text{for } t \in [T_0, T_n) \tag{47}$$

$$\widetilde{V}_x(t, \widetilde{b}(t)) = G_x(t, \widetilde{b}(t)) \qquad \text{for } t \in [T_0, T_n) \tag{48}$$

$$\widetilde{V}(t, x) < G(t, x) \qquad \text{in } \widetilde{C}^* \tag{49}$$

$$\widetilde{V}(t, x) = G(t, x) \qquad \text{in } \widetilde{D}^* \tag{50}$$

where the continuation set $\widetilde{C}^*$ and the stopping set $\widetilde{D}^*$ are given by

$$\widetilde{C}^* = \{\, (t, x) \in [T_0, T_n) \times (0, \infty) : x > \widetilde{b}(t) \,\} \tag{51}$$

$$\widetilde{D}^* = \{\, (t, x) \in [T_0, T_n) \times (0, \infty) : x \leq \widetilde{b}(t) \,\}. \tag{52}$$

American receiver swaption (4/5)

Theorem. The value function $\widetilde{V}$ has the following representation

$$\widetilde{V}(t,x) = \int_0^{T_n - t} \widetilde{L}(t,u,x,\widetilde{b}(t+u))du \tag{53}$$

for $t \in [T_0, T_n)$ and $x \in (0,\infty)$ and $\widetilde{b}$ can be characterized as the unique solution to the integral equation

$$G(t,\widetilde{b}(t)) = \int_0^{T_n - t} \widetilde{L}(t,u,\widetilde{b}(t),\widetilde{b}(t+u))du \tag{54}$$

for $t \in [T_0, T_n)$ in the class of continuous functions where

$$\widetilde{L}(t,u,x,z) = -\mathsf{E}H(t+u, X_u^x)I(X_u^x \le z) \tag{55}$$

for $t, u \ge 0$ and $x, z > 0$.

American receiver swaption (4/5)

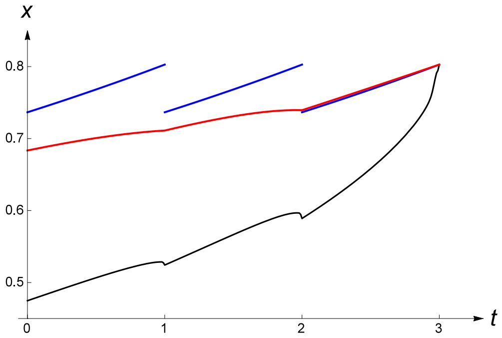

Figure 3. A computer drawing of the optimal stopping boundary $t \mapsto \widetilde{b}(t)$ (black). The parameter set is $T_0 = 0$, $\Delta = 1$ year, $n = 3$, $\theta = 2.66$, $\kappa = 0.03$, $\sigma = 0.29$, $K = 0.05$, $\alpha \equiv \theta\kappa = 0.08$.

Exercise strategy (1/4)

- The formulas above provide the prices of American swaptions for floating-rate receiver and fixed-rate receiver, respectively.
- However, the optimal stopping boundaries b and $\widetilde{b}$ provide the optimal exercise strategies in terms of process X, which is just the factor process of the SPD.
- Therefore our goal now is to connect the process X with some observable financial object and the natural choice is the *swap rate* of underlying swap contract.

Exercise strategy (2/4)

- The swap rate R_t is the fixed rate of swap above which makes $\Pi_t^{swap} = 0$ and hence we get the following relationship between R_t and X_t:

$$R_t = \frac{1 - P(t, T_n; X_t)}{(T_m - t)P(t, T_m; X_t) + \Delta \sum_{j=m+1}^{n} P(t, T_j; X_t)} =: f(t, X_t)$$

- The map $X_t \to P(t, T; X_t)$ is strictly decreasing in X_t and therefore using we have that $X_t \to R_t(X_t)$ is strictly increasing. Thus there is one-to-one relationship between R_t and X_t.

Exercise strategy (3/4)

The optimal exercise strategies in terms of swap rate R are given as follows

$$\tau_* = \inf \{ \ T_0 \leq s \leq T_n : R_s \geq r(s) \ \} \tag{56}$$

$$\widetilde{\tau}_* = \inf \{ \ T_0 \leq s \leq T_n : R_s \leq \widetilde{r}(s) \ \} \tag{57}$$

where the optimal exercise boundaries r and $\widetilde{r}$ are given as

$$r(t) = f(t, b(t)) \qquad \widetilde{r}(t) = f(t, \widetilde{b}(t)) \tag{58}$$

for $t \in [T_0, T_n]$.

Exercise strategy (4/4)

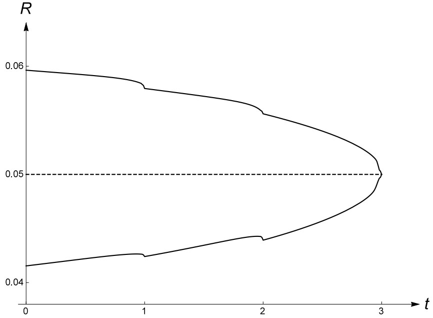

Figure 4. A computer drawing of the optimal exercise boundaries r (upper) and $\widetilde{r}$ (lower) in terms of the swap rate R_t. The parameter set is $T_0 = 0$, $\Delta = 1$ year, $n = 3$, $\theta = 2.66$, $\kappa = 0.03$, $\sigma = 0.29$, $K = 0.05$.

Callable bonds (1/2)

- Swaptions can be used to synthetically create callable bonds.
- Therefore the price of American swaption can be also interpreted as the value of including callable feature into the bond. This problem has its own independent interest.
- This argument also works for mortgage-backed securities, life insurance products etc.

Callable bonds (2/2)

Example. A company has issued a bond maturing in 10 years with annual coupons of 4%, and wants to add the option to call the bond at par at any time τ between $T_0 = 5$ and $T_n = 10$. If the company cannot change the original bond, they could buy a 5×5 American receiver swaption with strike rate 4%. This works as follows: suppose at time $\tau \in [T_0, T_n]$ the company decides to call the bond, that is, to exercise the American swaption. Clearly, the fixed coupon leg of the swap will then cancel the fixed coupon payments of the bond. On the other hand, paying the floating rate leg of the swap and the nominal N at maturity $T_n = 10$ is equivalent to paying the nominal N at τ, as desired.

Conclusion and future work

- The modelling of SPD by this framework allows us to formulate the American swaption problem as the undiscounted optimal stopping problem for one-dimensional diffusion process.
- We characterize the optimal stopping boundaries b and $\widetilde{b}$ as the unique solution to nonlinear integral equations and using this we obtain the arbitrage-free prices of the American swaptions and the optimal exercise strategies in terms of swap rates.
- Future work: multi-dimensional factor processes.
- Future work: different financial products, e.g. variance swaps, energy contracts, for other polynomial diffusion models.

Thank you!