Liquidity risk and optimal dividend/investment strategies

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Motivations : a corporate finance problem

Corporate governance problems : managerial decisions

1. **Operational levels** : marketing, product lines, internal organization...
2. **Financial levels** : capital/debt structure, type of financing...
3. **Cash-flow levels/utilization** : dividends and investments.

Motivations : a corporate finance problem

Cash-flow (utilization) policy

→ **Dividend policy : payment to shareholders**

- Share buyback (dividend or investment ?)
- Dividend payment

→ **Singular problem** : [1] Jeanblanc and Shiryaev (95), Choulli, Taksar and Zhou (03)

Motivations : a corporate finance problem

Cash-flow (utilization) policy

→ **Investment policy : investment for future growth**

- Organic growth : internal development of new products, technologies, factories...
- Merger/Acquisition : acquire a competitor for its product portfolio, geographic reach ...

→ **Optimal switching problem** : Brekke and Oksendal (94), Duckworth and Zervos (01), Hamadène and Jeanblanc (05), LV. and Pham (07), Pham, LV. and Zhou (09).

Combining both problems : Dividend and investment

Typical corporate dilemma : Total SA, ENI, BP : how to use the huge Cash-Flow generated ? which investment projects ?

Three observed policies :

1. Trader Classified Media : return over 90% of stock value in cash.
2. Bouygues : return 25% of stock value in cash.
3. Microsoft : almost no dividend payment for years !

Introduction - motivations : the Jeanblanc-Shiryaev problem.

A policy strategy is Z : a $\mathbb{F}$-adapted càdlàg non-decreasing process

We consider process X representing the dynamics of the cash reserve of a firm :

$$\begin{aligned} dX_t &= \mu dt + \sigma dW_t - dZ_t \\ X_{0-} &= x \end{aligned}$$

The value of the firm is defined as :

$$\hat{V}_0(x) = \sup_{Z \in \mathcal{Z}} \mathbb{E}\left[\int_0^{T_0^-} e^{-\rho t} dZ_t \right],$$

▷ $\hat{V}_0$ the value function of a singular problem

▷ characterized as the unique solution on $(0, \infty)$ with growth linear condition to $(V.I)_0$:

$$\min\left[\rho \hat{V}_0 - \mathcal{L}_0 \hat{V}_0 \, , \; \hat{V}_0' - 1\right] = 0, \quad x > 0, \tag{1}$$

$$\hat{V}_0(0) = 0. \tag{2}$$

Optimal dividend problem/singular control [1]

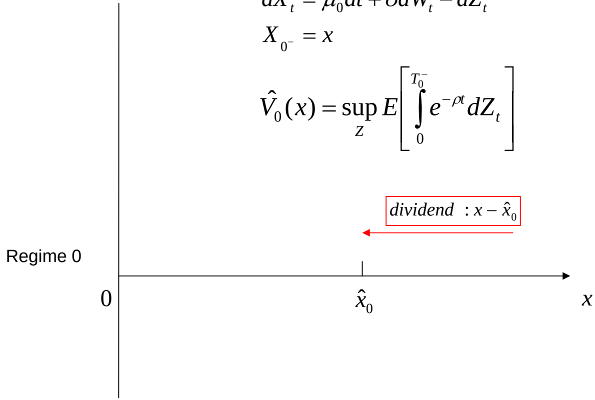
Explicit solution : Jeanblanc and Shiryaev (95) or Radner and Shepp (96)

▷ $\hat{V}_O$ explicit expression

$$dX_t = \mu_0 dt + \sigma dW_t - dZ_t$$

$$X_{0^-} = x$$

$$\hat{V}_0(x) = \sup_Z E\left[\int_0^{T_0^-} e^{-\rho t} dZ_t\right]$$



Benchmark 0: a singular control problem

Introduction - motivations : assumptions

In the previous problem, it is assumed :

- there is no investment opportunity
- the firm assets are infinitely liquid or illiquid

▶ Our objective : Study optimal dividend and investment control problem under constraints, i.e. relaxing the above assumptions

- L.V, Pham, and Villeneuve (2008) : the interaction between dividend policy and investment under uncertainty.
  - ▶ A mixte singular and switching control problem
  - ▶ Other related papers : Decamps and Villeneuve (05), Chevalier, L.V., Scotti (13), Guo and Tomecek (08)...
- In this study, we look at the dividend/investment problems but no longer assume that assets are infinitely illiquid or liquid. The firm may face some liquidity costs when buying or selling assets.

The model

- Let $(\Omega, \mathbb{F}, \mathbb{P})$ be a probability space equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ and W and B be two correlated $\mathbb{F}$-Brownian motions,

- A policy strategy is a singular/switching control $\alpha = ((\tau_i, q_i)_{i \in \mathbb{N}}, Z)$

  Z : $\mathbb{F}$-adapted càdlàg non-decreasing process representing dividend policy.

  (τ_i) : an increasing sequence of stopping times representing investment decision times.

  (q_i) : q_i are $\mathcal{F}_{\tau_i}$-measurable variables representing the number of productive assets units bought/sold.

- We assume productive assets are risky assets whose value process S is solution of the following equation :

$$dS_t = S_t \left(\mu dt + \sigma dB_t \right), \; S_0 = s, \tag{3}$$

The model

- the dynamics of the quantity of assets held by the firm $Q_t \in \mathbb{N}$ is governed by :

$$\left\{\begin{array}{lcl} dQ_t & = & 0 \ \text{ for } \tau_i \leq t < \tau_{i+1}, \\ Q_{\tau_i} & = & Q_{\tau_i^-} + q_i, \\ Q_0 & = & q, \end{array}\right. \qquad \text{for } i \in \mathbb{N}. \tag{4}$$

- The dynamics of the cash reserve (or more precisely the firm's cash and equivalents) process of the firm is governed by :

$$\left\{\begin{array}{lcl} dX_t & = & rX_t dt + h(Q_t)(bdt + \eta dW_t) - dZ_t, \text{ for } \tau_i \leq t < \tau_{i+1} \\ X_{\tau_i} & = & X_{\tau_i^-} - S_{\tau_i} f(q_i) q_i - \kappa, \\ X_0 & = & 0, \end{array}\right. \qquad \text{for } i \in \mathbb{N}. \tag{5}$$

where b, r and η are positive constants

h a non-negative, non-decreasing and concave function satisfying $h(q) \leq H$ with $h(1) > 0$ and $H > 0$.

The non-negative, non-decreasing, function f represents the liquidity cost function (or impact function with the impact being temporary) with $f(0) = 1$

Remarque. For a given fixed q, the model is closely related to the bachelier model used by Jeanblanc-Shiryaev.

The investment objective

- The bankruptcy time is defined as

$$T := T^{y,\alpha} := \inf\{t \geq 0, X_t < 0\}.$$

- We define the liquidation value as

$$L(x,s,q) := x + (sf(-q)q - \kappa)^+$$

- We introduce the following notation

$$\mathcal{S} := \mathbb{R}^+ \times (0,+\infty) \times \mathbb{N}.$$

- The optimal firm value is defined on $\mathcal{S} := \mathbb{R}^+ \times (0,+\infty) \times \mathbb{N}$, by

$$v(x,s,q) = \sup_{\alpha \in \mathcal{A}(x,s,q)} \mathbb{E}^{(x,s,q)} \left[\int_0^{T^-} e^{-\rho u} dZ_u \right] \tag{6}$$

Trivial case where value function is infinite

We now identify the trivial cases where the value function is infinite.

Lemma

If we have $r > \rho$ or $\mu > \rho$ then $v(y) = +\infty$ on $\mathcal{S}$.

Proof : Let $y := (x, s, q) \in \mathcal{S}$.

- We first assume that $\rho < r$.

At time 0, by choosing to liquidate the firm's assets, we may get $L(y) > 0$ in cash.

Then by waiting until a given time $t > 0$, we may obtain $v(y) \geq e^{(r-\rho)t} L(y)$.

By letting t going to $+\infty$, we have $v(y) = +\infty$.

- Assume that $\rho < \mu$.

First, suppose that $q \geq 1$.

In this case, by doing nothing up to time t and then liquidate at time t, for any $t > 0$,

we may obtain a lower bound of $v(y)$ which we prove goes to ∞ when t goes to ∞

Associated HJB equation

From the Dynamic Programming Principle, we may obtain the following HJB equation :

$$\min\{\rho\varphi(x,s,q) - \mathcal{L}\varphi(x,s,q); \frac{\partial\varphi}{\partial x}(x,s,q) - 1; v(x,s,q) - \mathcal{H}v(x,s,q)\} = 0 \text{ on } \mathcal{S}, \tag{7}$$

where we have set

$$\begin{aligned}
\mathcal{L}\varphi(x,s,q) &= \frac{\eta^2 h(q)^2}{2}\frac{\partial^2\varphi}{\partial x^2} + \frac{\sigma^2 s^2}{2}\frac{\partial^2\varphi}{\partial s^2} + c\sigma\eta s h(q)\frac{\partial^2\varphi}{\partial s\partial x} \\
&\quad + (rx + bh(q))\frac{\partial\varphi}{\partial x} + \mu s\frac{\partial\varphi}{\partial s} \\
\mathcal{H}\varphi(x,s,q) &= \max_{n\in\mathcal{A}(x,s,q)} \varphi(\Gamma(y,n)) \\
\text{with} \quad & \mathcal{A}(x,s,q) = \left\{n\in\mathbb{Z} : n \geq -q \text{ and } n(f(n)) \leq \frac{x}{s}\right\}, \\
& \Gamma(y,n) = (x - n(f(n))s,\ s,\ q+n).
\end{aligned}$$

and

$$\mathcal{S} = \{(x,s,q)\in\mathbb{R}^+ \times (0,+\infty)\times\mathbb{N}\}.$$

Sequence of auxiliary value functions

We now introduce the following subsets of $\mathcal{A}(y)$:

$$\mathcal{A}_N(y) \;:=\; \{\alpha \;=\; ((\tau_k, \xi_k)_{k \in \mathbb{N}^*}, Z) \in \mathcal{A}(y) : \tau_k \;=\; +\infty \text{ a.s. for all } k \geq N+1\}$$

We define

$$v_N(y) = \sup_{\alpha \in \mathcal{A}_N(y)} \mathbb{E}^{(x,s,q)} \left[\int_0^{T^-} e^{-\rho u} dZ_u \right], \; \forall N \in \mathbb{N} \tag{8}$$

The different steps to follow

- We characterize recursively the value functions v_N.
- We prove the convergence of v_N to v.
- We compute numerically the different regions (continuation, buy and sell and dividend regions) since no explicit solution may be obtained.

Sequence of auxiliary value functions

► In the next Proposition, we recall explicit formulas for v_0 and the optimal strategy associated to this singular control problem.

Proposition

There exists $x^*(q) \in [0, +\infty)$ such that

$$v_0(x, s, q) := \begin{cases} V_q(x) & \text{if} \quad 0 \leq x \leq x^*(q) \\ x - x^*(q) + V_q(x^*(q)) & \text{if} \quad x \geq x^*(q), \end{cases}$$

where V_q is the $\mathcal{C}^2$ function, solution of the following differential equation

$$\frac{\eta^2 h(q)^2}{2} y'' + (rx + bh(q))y' - \rho y = 0; \tag{9}$$

$$y(0) = 0, \;\; y'(x^*(q)) = 1 \text{ and } y''(x^*(q)) = 0. \tag{10}$$

Notice that $x \to v_0(x, s, q)$ is a concave and $\mathcal{C}^2$ function on $[0, +\infty)$ and that if $h(0) = 0$, it is optimal to immediately distribute dividends up to bankruptcy therefore $v_0(x, s, 0) = x$.

Auxiliary functions : Optimal stopping

► We now are able to characterize our impulse control problem as an optimal stopping time problem, defined through an induction on the number of interventions N.

Proposition

For all $(x, s, q, N) \in \mathcal{S} \times \mathbb{N}^*$, we have

$$v_N(x, s, q) = \sup_{(\tau, Z) \in \mathcal{T} \times \mathcal{Z}} \mathbb{E}\{\int_0^{T \wedge \tau} e^{-\rho u}\, dZ_u + e^{-\rho\tau} G_{N-1}(X^x_{\tau-}, S^s_\tau, q) 1_{\{\tau < T\}}\}, \quad (11)$$

where $\mathcal{T}$ is the set of stopping times, $\mathcal{Z}$ the set of predictable and non-decreasing càdlàg processes, and

$$G_{N-1}(x, s, q) := \max_{n \in a(x,s,q)} v_{N-1}(\Gamma(y, n)) \text{ and } G_{-1} = 0,$$

$$\text{with} \quad a(x, s, q) := \left\{ n \in \mathbb{Z} : n \geq -q \text{ and } nf(n) \leq \frac{x - \kappa}{s} \right\},$$

$$\text{and} \quad \Gamma(y, n) := (x - nf(n)s - \kappa,\ s,\ q + n).$$

Bounds and convergence of v_N

- We begin by stating a standard result which says that any smooth function, which is supersolution to the HJB equation, is a majorant of the value function.

Proposition

Let $N \in \mathbb{N}$ and $\phi = (\phi_q)_{q\in\mathbb{N}}$ be a family of non-negative $\mathcal{C}^2$ functions on $\mathbb{R}^+ \times (0, +\infty)$ such that $\forall q \in \mathbb{N}$ (we may use both notations $\phi(x, s, q) := \phi_q(x, s)$), $\phi_q(0, s) \geq 0$ for all $s \in (0, \infty)$ and

$$\min\left[\rho\phi(y) - \mathcal{L}^N\phi(y), \phi(y) - G_{N-1}(y), \frac{\partial\phi}{\partial x}(y) - 1\right] \quad \geq \quad 0 \tag{12}$$

for all $y \in (0, +\infty) \times (0, +\infty) \times \mathbb{N}$, where we have set

$$\begin{aligned}\mathcal{L}^N\varphi \quad &= \quad \frac{\eta^2 h(q)^2}{2}\frac{\partial^2\varphi}{\partial x^2} + (rx + bh(q))\frac{\partial\varphi}{\partial x} \\ &\quad +1_{\{N>0\}}\left[\frac{\sigma^2 s^2}{2}\frac{\partial^2\varphi}{\partial s^2} + c\sigma\eta s h(q)\frac{\partial^2\varphi}{\partial s\partial x} + \mu s\frac{\partial\varphi}{\partial s}\right].\end{aligned}$$

then we have $v_N \leq \phi$.

Corollary

Upper bound For all $N \in \mathbb{N}^*$ and $(x, s, q) \in \mathcal{S}$, we have

$$L(x, s, q) \leq v_N(x, s, q) \leq x + sq + K \quad \text{where } \rho K = bH.$$

Proposition(Convergence)

For all $y \in \mathcal{S}$, we have

$$\lim_{N \to +\infty} v_N(y) = v(y).$$

Proof : We obviously have $v_N \leq v_{N+1} \leq v$ for all $N \in \mathbb{N}$.
For $y \in \mathcal{S}$ and $\varepsilon > 0$, we may now consider a strategy $\alpha = ((\tau_k, \xi_k)_{k \in \mathbb{N}^*}, Z) \in \mathcal{A}(y)$ such that

$$v(y) \leq J^\alpha(y) + \varepsilon.$$

Notice that, as $(\tau_i)_{i \in \mathbb{N}^*}$ is such that $\lim_{i \to +\infty} \tau_i = +\infty$, there exists $N \in \mathbb{N}^*$ such that

$$\begin{aligned} J^\alpha(y) &\leq \mathbb{E}[\int_0^{T \wedge \tau_N} e^{-\rho s}\, dZ_s] + \varepsilon \\ &\leq v_N(y) + \varepsilon, \end{aligned}$$

which ends the proof.

Viscosity characterization of v_N

We turn to the characterization of the function v_N as the unique function which satisfies the boundary condition

$$v_N(y) = G_{N-1}(y) \text{ on } \{0\} \times (0, +\infty) \times \mathbb{N}. \tag{13}$$

and is a viscosity solution of the following HJB equation :

$$\min\{\rho v_N(y) - \mathcal{L} v_N(y); \frac{\partial v_N}{\partial x}(y) - 1; v_N(y) - G_{N-1}(y)\} = 0 \text{ on } (0, +\infty)^2 \times \mathbb{N}, \tag{14}$$

It relies on the following Dynamic Programming Principle.

DPP

Let $\theta \in \mathcal{T}$, $y := (x, s, q) \in \mathcal{S}$ and set $\nu = T \wedge \theta$, we have

$$v_N(y) = \sup_{(\tau, Z) \in \mathcal{T} \times \mathcal{Z}} \mathbb{E}[\int_0^{(\nu \wedge \tau)^-} e^{-\rho s}\, dZ_s + e^{-\rho(\nu \wedge \tau)} v_N\left(X^x_{(\nu \wedge \tau)^-}, S^s_{\nu \wedge \tau}, q\right) \mathbb{1}_{\{\tau < \nu\}}] \tag{15}$$

Viscosity characterization of v_N

- We are now able to establish the main results of this section.

Theorem

For all $(N, q) \in \mathbb{N}^* \times \mathbb{N}$, the value function $v_N(\cdot, \cdot, q)$ is continuous on $(0, +\infty)^2$. Moreover v_N is the unique viscosity solution on $(0, +\infty)^2 \times \mathbb{N}$ of the HJB equation (14) satisfying the boundary condition (13) and the following growth condition

$$|v_N(x, s, q)| \leq C_1 + C_2 x + C_3 sq, \quad \forall (x, s, q) \in \mathcal{S},$$

for some positive constants C_1, C_2 and C_3.

Numerical results

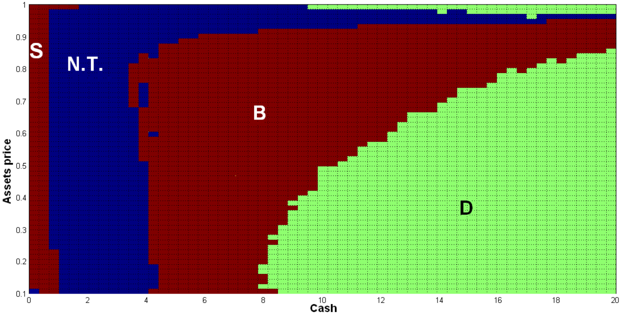


FIGURE: *Description of different regions, in (x, s) for a fixed q_0.*

Numerical results

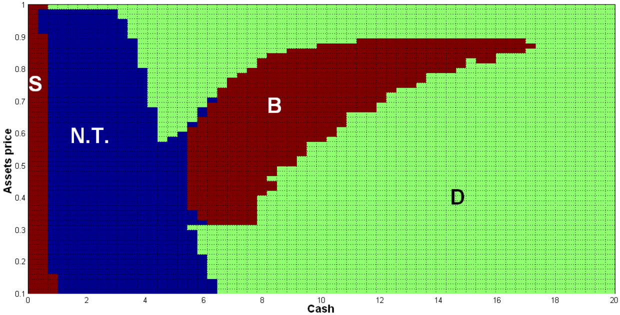


FIGURE: *Description of different regions, in* (x, s) *for* $q_1 > q_0$.