Financial Models with Defaultable Numéraires

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Lack of a natural numéraire

- **Classical models** of financial markets: prices in units of a **pre-specified numéraire**

$$(S_{1,1}(t) \equiv 1, S_{1,2}(t), \ldots, S_{1,d}(t))$$

- **Here:**
  1. **Absence of Numéraire:** Assets may potentially lose all value relative to the others

$$\mathbb{P}(S_{1,j}(t) = \infty) > 0$$

  2. **Symmetry of the assets:** Symmetrical point of view where all assets have equal priority

$$\frac{S_{1,j}(t)}{S_{1,i}(t)} \Rightarrow S_{i,j}(t)$$

Contribution

1. (**Formulation of the First and Second FTAP** - Symmetric as no asset is prioritized)
2. Interpretation of **strict local martingale models,**
   - Fix a numéraire that has positive probability to default
   - $\Rightarrow$ **Non-classical pricing formulas are economically justified and extended**
3. **Aggregation of non-equivalent pricing measures:**
   - Assume that for each asset there exists a probability measure under which discounted prices (with the corresponding asset as numéraire) are local martingales
   - These measures need not be equivalent
   - **Aggregatation of measures to an arbitrage-free pricing operator that takes all events of devaluations into account**

Non-classical pricing operators - Motivation

- Popular **model in FX:**

$$S_{1,2}(t) = S_{1,2}(0) + \int_0^t \left(aS_{1,2}(u)^2 + bS_{1,2}(u) + c\right) \mathrm{d}W(u)$$

"Quadratic normal volatility" (stopped when hitting zero)

- Calibration usually yields **strict local martingale dynamics**
- Let's assume a complete market
- **Superreplication cost of $S_{1,2}(T)$ is strictly smaller than** $S_{1,2}(0)$ (if we price according to risk-neutral expectations). This **contradicts no-arbitrage "in practice"**
- **Possible ways out:**
  - **Use a different model**
  - **Change the concept of pricing operator**

New pricing operators

- **Lewis (2000):** "add correction term" to risk-neutral expectation when pricing calls
- **Madan & Yor (2006):** Exchange expectations and limits
- **Cox & Hobson (2005):** Change class of admissible strategies
- **Paulot (2013):** Linear operator on a Banach space of payoffs
- **Carr & Fisher & Ruf (2014):**
  - A change of numéraire via strict local martingale $S_{1,2}$ yields non-equivalent measure
  - Consider the minimal superreplication cost under both measures (the original one and the new one)
  - Yields an explicit formula for the correction term

Issues:

- Correction term seems **non-symmetric in currencies**
- **What to do in an incomplete market?**
- **What to do with more than two currencies?**

Relative prices are modelled by an exchange matrix

- d: number of currencies
- Let $S_{i,j}(t)$ **denote the price of the j–th currency in terms of the i–th currency, at time t**
- $S = (S_{i,j})$ is an $\mathbb{F}$–progressive, càdlàg process taking values in $[0, \infty]^{d \times d}$ such that $S(t)$ **is an exchange matrix:**

$$S_{i,j}(t) S_{j,k}(t) = S_{i,k}(t) \quad \text{(whenever defined);}$$
$$S_{i,i}(t) = 1$$

- Note: there exists always a **strongest currency** i^* with $\sum_j S_{i*,j}(t) \leq d$
- $\mathfrak{A}(t) = \{i : \sum_j S_{i,j}(t) < \infty\} \neq \emptyset$ **(active currencies)**

Value vector

- A **value vector** $v = (v_i)$ (with respect to $S(t)$) encodes the price of an asset in terms of the d currencies.
- The i–th component describes the price of an asset in terms of the i–th currency.
- v satisfies consistency condition:

$$S_{i,j}(t)v_j = v_i \quad \text{(whenever defined)}.$$

- $\mathcal{D}^t$: the set of all $\mathcal{F}(t)$–**measurable value vectors with respect to** $S(t)$ (which are bounded, in a weak sense)

Valuation operator

- A **valuation operator** relates future random prices to present deterministic prices.
- Concept goes back to **Harrison & Pliska (1981); see also Biagini & Cont (2006)** and literature on risk measures.

$\mathbb{V} = (\mathbb{V}^{r,t})_{0 \leq r \leq t \leq T}$, with

$$\mathbb{V}^{r,t} : \mathcal{D}^t \to \mathcal{D}^r,$$

is a valuation operator with respect to S if it satisfies:

1. Positivity
2. Linearity
3. Continuity from below
4. Time consistency
5. Martingale property
6. Redundancy

Valuation operator — the conditions

1. **(Positivity)** If $C \in \mathcal{D}^T$ and $C \geq 0$ then $\mathbb{V}^{0,T}(C) \geq 0$
2. **(Linearity)** If $H \in \mathcal{L}^{\infty,r}$, and $C, C' \in \mathcal{D}^t$ then

$$\mathbb{V}^{r,t}(H\mathbf{1}_{\{H\neq 0\}}C + C') = H\mathbf{1}_{\{H\neq 0\}}\mathbb{V}^{r,t}(C) + \mathbb{V}^{r,t}(C')$$

3. **(Continuity from below)** If $C_n \uparrow C \in \mathcal{D}^T$, then $\mathbb{V}^{0,t}(C_n) \uparrow \mathbb{V}^{0,t}(C)$
4. **(Time consistency)** For $C \in \mathcal{D}^T$,

$$\mathbb{V}^{r,t}(\mathbb{V}^{t,T}(C)) = \mathbb{V}^{r,T}(C)$$

5. **(Martingale property)** $\mathbb{V}^{t,T}(S_{\cdot,i}(T)) = S_{\cdot,i}(t)\mathbf{1}_{\{i\in\mathfrak{A}(t)\}}$
6. **(Redundancy)** For $C \in \mathcal{D}^t$ with $\sum_i \mathbf{1}_{\{C_i=0\}} > 0$, $\mathbb{V}^{r,t}(C) = 0$

Disaggregation and aggregation

$(\mathbb{Q}_i)$ is a **consistent family of probability measures** if

- S_i a $\mathbb{Q}_i$–supermartingale
- Change-of-numéraire formula holds:

$$\mathbb{E}^{\mathbb{Q}_i}[S_{i,j}(t)\mathbf{1}_A] = S_{i,j}(0) \times \mathbb{Q}_j(A \cap \{S_{j,i}(t) > 0\})$$

Theorem:

- **(Disaggregation)** Given a valuation operator $\mathbb{V}$ there exists a consistent family of supermartingale measures $(\mathbb{Q}_i)$ such that

$$\mathbb{V}_j^{r,t}(C) = \sum_i S_{j,i}(r)\mathbb{E}_r^{\mathbb{Q}_i}\left[\frac{C_i}{|\mathfrak{A}(t)|}\right] \tag{1}$$

  for all $r \leq t$, $j \in \mathfrak{A}(r)$, $C \in \mathcal{D}^t$
- **(Aggregation)** Conversely, given a consistent family of supermartingale measures $(\mathbb{Q}_i)$, (1) defines a valuation operator $\mathbb{V}$

The appearance of strict local martingales

Consistent family $(\mathbb{Q}_i)$, with $A = \Omega$:

$$\mathbb{E}^{\mathbb{Q}_i}[S_{i,j}(t)] = S_{i,j}(0) \times \mathbb{Q}_j(S_{j,i}(t) > 0)$$

- $S_{i,j}$ **is a** $\mathbb{Q}_i$**–martingale if and only if** $\mathbb{Q}_j(S_{j,i}(T) = 0) = 0$
- $S_{i,j}$ **is a** $\mathbb{Q}_i$**–local martingale if and only if** $S_{j,i}$ **does not jump to zero under** $\mathbb{Q}_j$

The case of two assets - Call option

$d = 2$, with value vector $C = (C_1, C_2)^{\mathrm{T}}$

$C = ((S_{1,2}(T) - K)^+, (1 - KS_{2,1}(T))^+)^{\mathrm{T}}$

$$\begin{aligned}\mathbb{V}_j^{0,T}(C) &= S_{j,1}(0) \times \mathbb{E}^{\mathbb{Q}_1}\left[\frac{C_1}{|\mathfrak{A}(T)|}\right] + S_{j,2}(0) \times \mathbb{E}^{\mathbb{Q}_2}\left[\frac{C_2}{|\mathfrak{A}(T)|}\right] \\ &= S_{j,1}(0) \times \mathbb{E}^{\mathbb{Q}_1}[C_1] + \underbrace{S_{j,2}(0) \times \mathbb{E}^{\mathbb{Q}_2}[C_2 \mathbf{1}_{\{S_{1,2}(T)=\infty\}}]}_{\text{Lewis' correction term}}\end{aligned}$$

Example: The Camara-Heston model

- **Câmara & Heston (2008): Extension of BSM model with a huge jump upward or a huge jump downward to explain observed skews and smiles.**
- They derive analytic call and put prices by solving a suitable PDE.
- In our setup: $d = 2$
- W is $\mathbb{P}$–BM, and τ_1, τ_2 are independent exponential times with intensities λ_1, λ_2:

$$S_{1,2}(t) = \mathrm{e}^{\sigma W(t)+\mu t}\mathbf{1}_{\{t \leq \tau_1 \wedge \tau_2\}} + \infty \times \mathbf{1}_{\{\tau_1 < \tau_2 \wedge t\}}$$

- **Call option with $\mathbf{C_1 = (S_{1,2}(T) - K)^+}$ and $\mathbf{C_2 = (1 - KS_{2,1}(T))^+}$.** Then

$$\mathbb{V}_1^{0,T}(C) = \mathrm{e}^{-\lambda_1 T} S_{1,2}(0)\Phi(d_1) - K\mathrm{e}^{-\lambda_2 T}\Phi(d_2) + \underbrace{S_{1,2}(0)(1 - \mathrm{e}^{-\lambda_1 T})}_{\text{correction term}}$$

The concept of "no obvious devaluations"

We say that a probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F}(T))$ satisfies **"No Obvious Devaluations" (NOD)** if

$$\mathbb{P}(i \in \mathfrak{A}(T) | \mathcal{F}(\tau)) > 0 \text{ on } \{\tau < \infty\} \cap \{i \in \mathfrak{A}(\tau)\}$$

for all i and stopping times τ

Aggregation without numéraire-consistency

Let $(\mathbb{Q}_i)$ be be a family of probability measures. Then **there exists a martingale valuation operator** $\mathbb{V} \sim \sum_i \mathbb{Q}_i$ if one of the following two conditions is satisfied:

1. $\mathbf{S_i}$ **is a** $\mathbb{Q}_\mathbf{i}$**–martingale for all** $\mathbf{i}$
2. The following four conditions hold:
   - 2.1 $\mathbf{S_i}$ **is a** $\mathbb{Q}_\mathbf{i}$**–local martingale for all** $\mathbf{i}$
   - 2.2 $\sum_\mathbf{i} \mathbb{Q}_\mathbf{i}$ **satisfies (NOD)**
   - 2.3
   $$\mathbb{Q}_k|_{\mathcal{F}\cap\{\sum_j S_{k,j}(T)<\infty\}} \sim \left(\sum_i \mathbb{Q}_i\right)\Bigg|_{\mathcal{F}\cap\{\sum_j S_{k,j}(T)<\infty\}}$$
   - 2.4 There exist $\epsilon > 0$, $N \in \mathbb{N}$, predictable times $(T_n)_{n\in\{1,\cdots,N\}}$ s.t.
   $$\bigcup_k \left\{(t,\omega) : \sum_j S_{k,j}(t) = \infty \text{ and } \sum_j S_{k,j}(t-) \leq d + \varepsilon\right\} \subset \bigcup_{n=1}^N [\![T_n]\!]$$

Conclusion

- We consider an exchange economy with d currencies, where **each currency has the possibility to completely devaluate against any other currency**
- (In such an economy, we introduce the concept of a **valuation operator and link it to a no-arbitrage condition**)
- We interpret the **lack of martingale property** of an asset price as a reflection of the **possibility that the numéraire currency may devaluate completely**
- We study **conditions under which not necessarily equivalent measures**, corresponding to different numéraires, **may be aggregated** to obtain a numéraire-independent valuation operator

Muchas gracias!

Example: Multi-currency market

- d-**currencies:** any currency can devaluate completely with respect to any other currency
- (τ_i): Times of devaluation
- **Relative prices:**

$$S_{i,j}(\cdot) = \mathrm{e}^{B_{i,j}(\cdot)} \mathbf{1}_{[\![0,\tau_j[\![} + \infty \mathbf{1}_{\{\tau_i < \tau_j\}} \mathbf{1}_{[\![\tau_i,\infty[\![}$$

where $(B_{i,j})$ are of FV

- **Consistency of family $(\mathbb{Q}_i)$:** compensator of τ_j w.r.t. $\mathbb{Q}_i$ is $A_{i,j} = A_j \mathbf{1}_{i \neq j}$ for predictable processes of the form

$$A_i(\cdot) = \int_0^{\cdot} \lambda_i(s)\, ds$$

such that

$$B_{i,j} = A_j - A_i. \qquad \text{on } \{\tau_i \wedge \tau_j > t\}$$

Example: Multi-currency market (cont.)

- For example, in the same spirit of **Jarrow & Yu (2001) and Collin-Dufresne et. al. (2004),** suppose that $B_{i,j} = 0$ and

$$A_i(t) = \left(\lambda_b + (\lambda_a - \lambda_b)\mathbf{1}_{\{t > \min_j \tau_j\}}\right)(t \wedge \tau_i)$$

for all i, with $\lambda_b, \lambda_a > 0$

- Consider an **exchange option:** $C_i = (S_{i,2}(T) - KS_{i,1}(T))^+$. If $K \in [0,1]$ and $\lambda_a \neq (d-1)\lambda_b$ then

$$\begin{aligned}
\mathbb{V}_2^{0,T}(C) &= \mathbb{Q}_2(\tau_1 \leq T) + (1-K)\mathbb{Q}_2(\tau_1 > T) \\
&= 1 - K\mathbb{Q}_2(\tau_1 > T) \\
&= 1 + \frac{K}{(\lambda_a - (d-1)\lambda_b)}\left((\lambda_b - \lambda_a)\mathrm{e}^{-(d-1)\lambda_b T} + (d-2)\lambda_b \mathrm{e}^{-\lambda_a T}\right)
\end{aligned}$$

Digression: Hyperinflation

- *Hyperinflation*: complete devaluation of the corresponding domestic numéraire and an explosion of the exchange rate with respect to any other currency
- Examples:
  - The price of one Dollar, measured in units of the respective domestic currency, went up by a factor of over 4500 in Austria from January 1919 to August 1922 and by a factor of over 10^{10} from January 1922 to December 1923 in Germany
  - Hungary, August 1945 to July 1946. Prices soared by a factor of over 10^{27} in that 12-month period to which the month of July contributed a staggering raise of $4 * 10^{16}$ percent of prices
  - Bolivia, August 1984 to August 1985: Price levels increased by $20,000$ percent
  - Zimbabwe, July 2009: for instance, prices increased by an annualized inflation rate of over $2 * 10^8$ percent

Related literature — an incomplete list

- Herdegen & Schweizer (2015)
- Herdegen (2014)
- Tehranchi (2014): Non-existence of numéraire
- Kardaras (2014): Exchange options
- Carr & Fisher & Ruf (2014)
- Schönbucher's survival measure (credit risk)
- Yan (1998): Basket numéraire
- Delbaen & Schachermayer (199x): FTAP, changes of numéraires, ...
- ...

Introduction of a probability measure

- Let $\mathbb{P}$ be a probability measure on $(\Omega, \mathcal{F})$
- We say $\mathbb{P}$ satisfies (PSmg) if there exists (A_i) with $\bigcup_i A_i = \Omega$ such that for each i, $\mathbb{P}(A_i) > 0$ and S_i is a $\mathbb{P}_i$–semimartingale, where $\mathbb{P}_i(\cdot) = \mathbb{P}(\cdot|A_i)$ for each i

Trading strategies and wealth processes

- Let $\mathbb{P}$ satisfy (PSmg)
- Let h denote a predictable process. Then V^h is a value vector process with $V_i^h(t) = \sum_j h_j S_{i,j}(t)$
- h is called a $\mathbb{P}$–*trading strategy* if $h \in L(S_i, \mathbb{P}_i)$ and the *self-financing condition holds*:

$$V_i^h - V_i^h(0) = h \cdot_{\mathbb{P}_i} S_i$$

- h is $\mathbb{P}$–*allowable* if there exists $\varepsilon > 0$ such that $V_i(t) \geq -\varepsilon \sum_j S_{i,j}(t)$

No-arbitrage condition

Assume that $\mathbb{P}$ satisfies (PSmg).
We say that S satisfies **NFLVR for $\mathbb{P}$–allowable strategies** if for any sequence of $\mathbb{P}$–allowable strategies (h^n) with $V^{h^n}(0) \leq 0$ and such that there exist $(\xi^n) \in L^\infty(\mathbb{R}, \mathbb{P})$ satisfying

$$V_i^{h^n}(T) \geq \xi^n \sum_j S_{i,j}(T),$$

the following conclusion holds:

$$\xi = \lim_{n\uparrow\infty} \xi^n \text{ exists and } \mathbb{P}(\xi \geq 0) = 1 \quad \Longrightarrow \quad \mathbb{P}(\xi = 0) = 1.$$

Here, the limit is taken in $L^\infty(\mathbb{R}, \mathbb{P})$

First FTAP as a bridge between the mathematics and the finance

"absence of arbitrage"	martingale measure
pricing	expectation
NFLVR for allowable strategies	**valuation operator**

First fundamental theorem

Write $\mathbb{P} \sim \mathbb{V}$ if for a nonnegative $C = (C_i) \in \mathcal{D}^T$, we have $\mathbb{V}^{0,T}(C) = 0$ if and only if $\sum_i \mathbf{1}_{\{C_i=0\}} > 0$ $\mathbb{P}$–almost surely.

1. If $\mathbb{P}$ satisfies (PSmg) and S satisfies NFLVR for $\mathbb{P}$–allowable strategies then there exists a valuation operator $\mathbb{V} \sim \mathbb{P}$.
2. If there exists a valuation operator $\mathbb{V}$ then there exists a probability measure $\mathbb{P} \sim \mathbb{V}$ that satisfies (PSmg) and such that S satisfies NFLVR for $\mathbb{P}$–allowable strategies.

Second fundamental theorem

Suppose that there exists a valuation operator $\mathbb{V}$ with respect to S. Then, the market is complete if and only if $\mathbb{V}$ is the unique valuation operator equivalent to $\mathbb{V}$.

Moreover, if a valuation operator exists, then

$$\inf\{V^h(0) : h \text{ super-replicates } C\} = \sup\{\widetilde{\mathbb{V}}^{0,T}(C) : \widetilde{\mathbb{V}} \sim \mathbb{V} \text{ is a valuation operator}\},$$

Furthermore, the infimum is obtained if the above expression is finite.