Utility maximization in a multi-dimensional semi-martingale setting with nonlinear wealth dynamics

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Outline

1. Introduction
2. Market model
3. Convex duality approach
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Expected utility maximization problem

Market model

$X = (X_t)_{t\geq 0}$ semi-martingale in $\mathbb{R}^d$ defined on a filtered prob. space $(\Omega, \mathscr{F}, \mathbb{F}, \mathbf{P})$

Financial market consists of d risky assets and one locally risk-free asset

$$S_t^i = \exp(X_t^i), \quad i = 1, \ldots d \qquad\qquad S_t^0 = \exp\left(\int_0^t r_s\, ds\right), \quad t \geq 0$$

Goal maximize $J(x; \pi, \gamma) = \mathbb{E}\left[\int_0^T U_1(t, \gamma_t)\, dt + U_2(V_T^{x,\pi,\gamma})\right]$

subject to

$$dV_t = V_{t-}\left\{ g(t, \pi_t)\, dt + \sum_{i=1}^d \pi_t^i \frac{dS_t^i}{S_{t-}^i} + (1 - \pi_t \cdot \underline{1}) \frac{dS_t^0}{S_{t-}^0} \right\} - \gamma_t\, dt, \quad V_0 = x.$$

$\pi_t^i =$ proportion of wealth invested in risky asset S_t^i at time $t \in [0, T], \ i = 1, \ldots, d$

$\gamma_t =$ instantaneous consumption rate

$K \subset \mathbb{R}^d$ non-empty, closed and convex constraint set with $0 \in K$

$g(t, \cdot) : K \to \mathbb{R}$ concave penalty (margin payment) function with $g(t, 0) = 0$

Two examples

Example 1. Borrowing interest rate R_t higher than lending rate r_t

$$g(t,\pi) := -(R_t - r_t)(\pi - 1)^+, \quad \pi \in \mathbb{R}.$$

Example 2. Stock loan fee r_t^L higher than margin account interest rate r_t^M (negative rebate rate)

$$g(t,\pi) = (r_t^M - r_t^L)\pi^-.$$

Here $\pi^- := -\min\{0,\pi\}$ denotes the negative part of $\pi \in \mathbb{R}$.

Key points

1. Use simple convex analysis tools to adapt classical martingale method to solve the utility maximization problem for non-linear wealth dynamics
2. Obtain sufficient conditions for existence of optimal investment consumption strategies in a multi-dimensional exponential semi-martingale model
3. Explicit characterization for CRRA utility, differential borrowing and lending rates and short-sale with negative rebates rates

Related works

- Domenico Cuoco and Hong Liu, *Math. Finance* 10 (2000), no. 3, 355–385.
- I. Klein and L. C. G. Rogers, *Math. Finance* 17 (2007), no. 2, 225–247.
- C. Labbé and A. Heunis *Adv. Appl. Prob.* 39 (2007) 77–104
- A. Heunis *SIAM J. Control Opt.* 53 (2015), no. 4, 2608–2656.

Assumptions

1. X has canonical representation

$$X_t = X_0 + B_t + X_t^{\mathsf{c}} + h(y) * (\mu - \rho) + [y - h(y)] * \mu$$

with predictable local characteristics (b, a, ν)

$$B_t^i = \int_0^t b_s^i \, ds, \quad \left\langle X^{i,\mathsf{c}}, X^{j,\mathsf{c}} \right\rangle_t = \int_0^t a_s^{ij} \, ds, \quad \rho(dy, dt) = \nu_t(dy) \, dt$$

Recall $\nu = (\nu_t)_{t \geq 0}$ is a transition kernel from $(\Omega \times \mathbb{R}_+; \mathscr{P})$ into $(\mathbb{R}^d; \mathscr{B}(\mathbb{R}^d))$ with

$$\int_0^t \int_{\mathbb{R}^d} (|y|^2 \wedge 1) \, \nu_s(dy) \, ds < +\infty, \quad \text{for all } t \geq 0.$$

2. X^{c} and $\tilde{\mu} = \mu - \rho$ have the weak representation property
   Example 1. Brownian motion + (independent) marked point process
   Example 2. Levy process.

3. $U_1(t, \cdot)$ and U_2 concave $\mathscr{C}^1$ utility functions satisfying Inada condition
   $U_1'(t, 0) = U_2'(0) = +\infty$

Martingale method + convex duality: $g = 0$ and $K = \mathbb{R}^d$

Let Θ denote the set of pairs $\varphi = (\varphi^1, \varphi^2)$ satisfying

$$0 = r_t \underline{1} - b_t - \frac{1}{2}\mathrm{diag}[a_t] - a_t \varphi_t^1 - \int_{\mathbb{R}^d} \Big[(e^y - \underline{1}) \varphi^2(t,y) - h(y) \Big] \, \nu_t(dy), \ \ \forall t \in [0,T].$$

For each $\varphi \in \Theta$, let H^φ denote the adjoint process

$$dH_t = H_{t-} \left\{ -r_t \, dt + \varphi_t^1 \cdot dX_t^{\mathsf{c}} + \int_{\mathbb{R}^d} (\varphi^2(t,y) - 1) \, \tilde{\mu}(dy,dt) \right\}, \ \ H_0 = 1.$$

Using Ito's formula

$$H_t^\varphi V_t^{x,\pi,\gamma} + \int_0^t H_s^\varphi \gamma_s \, ds \le x + \int_0^t (\cdots) \cdot dX_t^{\mathsf{c}} + \int_0^t \int_{\mathbb{R}^d} (\cdots) \, \tilde{\mu}(dy,dt), \ \ t \in [0,T].$$

Budget constraint holds

$$\mathbb{E}\left[H_T^\varphi V_T^{x,\pi} + \int_0^T H_t^\varphi \gamma_t \, dt \right] = x, \ \ \forall \varphi \in \Theta, \ \ \forall (\pi,\gamma) \in \mathscr{A}(x).$$

Martingale method + convex duality: $g = 0$ and $K = \mathbb{R}^d$

For simplicity, take $\gamma_t = 0$.

Let $I := (U')^{-1}$ be the inverse marginal utility (from final wealth). Then

$$U(x) \le U(I(y)) + y(x - I(y)), \quad \text{for all } x > 0,\ y > 0.$$

Using this together with budget constraint $\mathbb{E}\left[H_T^{\varphi} V_T^{x,\pi}\right] \le x$ we get

$$\begin{aligned}\mathbb{E}[U(V_T^{x,\pi})] &\le \mathbb{E}[U(I(yH_T^{\varphi}))] + \left\{\mathbb{E}[yH_T^{\varphi}(V_T^{x,\pi} - I(yH_T^{\varphi}))]\right\} \\ &\le \mathbb{E}[U(I(yH_T^{\varphi}))] + y\left\{x - \mathbb{E}[H_T^{\varphi} I(yH_T^{\varphi})]\right\}\end{aligned}$$

It can be shown that the map

$$\mathscr{X}^{\varphi}(y) := \mathbb{E}\left[H_T^{\varphi} I(yH_T^{\varphi})\right]$$

is invertible. Define $\mathscr{Y}^{\varphi}(x) := \left(\mathscr{X}^{\varphi}\right)^{-1}(x)$ and

$$G^{x,\varphi} := I(\mathscr{Y}^{\varphi}(x) H_T^{\varphi}).$$

It follows

$$\mathbb{E}[U(V_T^{x,\pi})] \le \mathbb{E}[U(G^{x,\varphi})], \quad \text{for all } \pi \in \mathscr{A}(x),\ \varphi \in \Theta.$$

Martingale method + convex duality: general case

Let δ_K be the indicator function (in the sense of convex analysis) of the portfolio constraint set K

$$\delta_K(\pi) := \begin{cases} 0, & \text{if } \pi \in K \\ +\infty, & \text{if } \pi \notin K \end{cases}$$

Define $g_K(\omega,t,\pi) := g(\omega,t,\pi) - \delta_K(\pi)$ and

$$\tilde{g}_K(\omega,t,\zeta) := \sup_{\pi\in\mathbb{R}} [g_K(\omega,t,-\pi) + \pi\zeta] = \sup_{\pi\in K} [g(\omega,t,\pi) - \pi\zeta], \;\; \zeta \in \mathbb{R}$$

as the convex conjugate of $\mathbb{R} \ni \pi \mapsto -g_K(t,-\pi) \in \mathbb{R}$.

- Effective domain of $\tilde{g}_K(\omega,t,\cdot)$
$$\mathscr{N}_t(\omega) := \{\zeta \in \mathbb{R} : \tilde{g}_K(\omega,t,\zeta) < +\infty\}$$
- $\mathscr{D}$:= class of progressively measurable processes $\{\zeta_t\}_{t\in[0,T]}$ satisfying
$$\sup_{t\in[0,T]} |\zeta_t| + \int_0^T \tilde{g}_K(t,\zeta_t)\,dt < +\infty, \;\; \text{a.s.}$$

Martingale method + convex duality: general case

Let now Θ denote the set of pairs (φ^1, φ^2) for which the process

$$\zeta_t^{\varphi} := r_t \underline{1} - b_t - \frac{1}{2}\mathrm{diag}[a_t] - a_t \varphi_t^1 - \int_{\mathbb{R}^d} \left[(e^y - \underline{1})\varphi^2(t,y) - h(y) \right] \nu_t(dy), \quad t \in [0,T]$$

belongs to $\mathscr{D}$.

For each $\varphi \in \Theta$, let H^{φ} be the solution of the linear SDE

$$dH_t = H_{t-} \left\{ -[r_t + \tilde{g}_K(t, \zeta_t^{\varphi})]\, dt + \varphi_t^1 \cdot dX_t^{\mathsf{c}} + \int_{\mathbb{R}^d} [\varphi^2(t,y) - 1]\, \tilde{\mu}(dy,dt) \right\}$$

$$H_0 = 1.$$

Then budget constraint holds

$$\mathbb{E}\left[H_T^{\varphi} V_T^{x,\pi} + \int_0^T H_t^{\varphi} \gamma_t \, dt \right] \leq x, \quad \forall \varphi \in \Theta, \quad \forall (\pi, \gamma) \in \mathscr{A}(x)$$

also holds.

Martingale method + convex duality: general case

For each $\varphi \in \Theta$ define the map

$$\mathscr{X}^{\varphi}(y) := \mathbb{E}\left[\int_0^T H_t^{\varphi} I_1(t, yH_t^{\varphi})\,dt + H_T^{\varphi} I_2(yH_T^{\varphi})\right].$$

Let $\widetilde{\Theta} := \{\varphi \in \Theta : \mathscr{X}^{\varphi}(y) < \infty,\ \forall y > 0\}$.

For each $\varphi \in \widetilde{\Theta}$ define $\mathscr{Y}^{\varphi} := (\mathscr{X}^{\varphi})^{-1}$ and

$$\begin{aligned} \gamma_t^{x,\varphi} &:= I_1(t, \mathscr{Y}^{\varphi}(x) H_t^{\varphi}),\ t \in [0,T], \\ G^{x,\varphi} &:= I_2(\mathscr{Y}^{\varphi}(x) H_T^{\varphi}). \end{aligned}$$

Then, the auxiliary functional

$$L(x;\varphi) := \mathbb{E}\left[\int_0^T U_1(t, \gamma_t^{x,\varphi})\,dt + U_2(G^{x,\varphi})\right],\ \ x > 0,\ \varphi \in \widetilde{\Theta}.$$

satisfies

$$J(x;\pi,\gamma) \le L(x;\varphi),\ \forall (\pi,\gamma) \in \tilde{\mathscr{A}}(x),\ \forall \varphi \in \widetilde{\Theta}.$$

Main result

For each $\varphi \in \tilde{\Theta}$ let the triple $(Y^{x,\varphi}, \alpha^{x,\varphi}, \beta^{x,\varphi})$ be the solution to the linear backward SDE

$$Y_t^{x,\varphi} = H_T^{\varphi} G^{x,\varphi} + \int_t^T H_s^{\varphi} \gamma_s^{x,\varphi}\, ds - \int_t^T \alpha_s^{x,\varphi}\, dX_s^{\mathsf{c}} - \int_t^T \int_{\mathbb{R}^d} \beta^{x,\varphi}(s,y)\, \tilde{\mu}(dy,ds).$$

Theorem For $x > 0$ fixed, suppose there exists $\hat{\varphi} \in \tilde{\Theta}$ and $\hat{\pi}_t$ satisfying

(i) $\hat{\pi}_t := \dfrac{\alpha_t^{x,\hat{\varphi}}}{Y_{t-}^{x,\hat{\varphi}}} - \varphi_t^1 \in K$, for all $t \in [0,T]$

(ii) $1 + \hat{\pi}_t \cdot (e^y - \underline{1}) = \dfrac{1}{\hat{\varphi}^2(t,y)} \left[\dfrac{\beta^{x,\hat{\varphi}}(t,y)}{Y_{t-}^{x,\hat{\varphi}}} + 1 \right]$, for ρ-a.e. $(t,y) \in [0,T] \times \mathbb{R}^d$

(iii) $g(t,\hat{\pi}_t) - \hat{\pi}_t \zeta_t^{\hat{\varphi}} = \tilde{g}_K(t, \zeta_t^{\hat{\varphi}})$, for all $t \in [0,T]$.

Then $(\hat{\pi}, \hat{\gamma}) \in \tilde{\mathscr{A}}(x)$ with $\hat{\gamma} = \gamma^{x,\hat{\varphi}}$ is optimal.

Log utility $U_1(t,x) = U_2(x) = \ln x$

For log-utility, it can be proved easily that for all $\phi \in \tilde{\Theta}$ we have

$$\alpha_t^{x,\varphi} = 0 \quad \text{and} \quad \beta^{x,\varphi}(t,y) = 0, \;\; \text{a.s. for } \rho\text{-a.e. } (t,y) \in [0,T] \times \mathbb{R}^d.$$

Then, optimality conditions (i)-(ii) reduce to

$$\varphi_t^1 = -\pi_t \quad \text{and} \quad \varphi^2(t,y) = \frac{1}{1 + \hat{\pi}_t \cdot (e^y - \underline{1})}.$$

In what follows, suppose further that jumps of X have bounded variation

$$\int_{\mathbb{R}^d} h(y)\, \nu_t(dy) = \int_{|y|\le 1} y\, \nu_t(dy)\, dt < +\infty, \quad \forall t \in [0,T]$$

Example. Generalized tempered stable jumps in 1-dim

$$\nu_t(dy) = \frac{c_t^1 e^{-\lambda_t^1 y}}{y^{1+d_t^1}} \mathbf{1}_{\{y>0\}}\, dy + \frac{c_t^2 e^{-\lambda_t^2 |y|}}{|y|^{1+d_t^2}} \mathbf{1}_{\{y<0\}}\, dy$$

with $d_t^i < 1$, $c_t^i > 0$ and $\lambda_t^i \ge 0$, $i = 1,2$.

Log utility $U_1(t,x) = U_2(x) = \ln x$

Define for each $t \in [0,T]$

$$q_t(\pi) := r_t \underline{1} - b_t - \frac{1}{2}\mathrm{diag}[a_t] + a_t\pi_t - \int_{\mathbb{R}^d} \left[\frac{1}{1+\pi\cdot(e^y - \underline{1})}(e^y - \underline{1}) - h(y) \right] \nu_t(dy)$$

Theorem Let $\hat{\pi} = (\hat{\pi}_t)_{t\in[0,T]}$ be a K-valued predictable portfolio process satisfying

- $1 + \hat{\pi}_t \cdot (e^y - \underline{1}) > 0$, a.s. for ρ-a.e. $(t,y) \in [0,T] \times \mathbb{R}^d$,
- $\{q_t(\hat{\pi}_t),\ t \in [0,T]\}$ belongs to $\mathscr{D}$ and

$$g(t,\hat{\pi}_t) - \hat{\pi}_t q_t(\hat{\pi}_t) = \tilde{g}_K(t, q_t(\pi_t)) \ \text{ a.s. for all } t \in [0,T].$$

Then the pair $(\hat{\pi},\hat{\gamma})$ is optimal, where $\hat{\gamma} = (\hat{\gamma}_t)_{t\in[0,T]}$ is the consumption process defined by $\hat{\gamma}_t := \frac{x}{(1+T)} V_t^{1,\hat{\pi},0}$, $t \in [0,T]$.

Moreover, the optimal wealth process $V^{x,\hat{\pi},\hat{\gamma}}$ satisfies

$$V_t^{x,\hat{\pi},\hat{\gamma}} = V_t^{x,\hat{\pi},0} - t\hat{\gamma}_t = V_t^{x,\hat{\pi},0}\left(1 - \frac{t}{T+1}\right), \quad t \in [0,T].$$

Power utility $U_1(t,x) = U_2(x) = \frac{x^p}{p}$ with $p < 1, p \neq 0$

Assume further

- local characteristics (b, a, ν), the force of interest r_t and the margin payment function $g(t, \pi)$ are non-random
- for simplicity, consumption is not allowed

Under these conditions, for power utility the following holds

Lemma For all $\varphi \in \tilde{\Theta}$ we have

$$\frac{\alpha_t^{x,\varphi}}{Y_{t-}^{x,\varphi}} = \frac{-p}{p-1}\varphi_t^1 \quad \text{and} \quad \frac{\beta^{x,\varphi}(t,y)}{Y_{t-}^{x,\varphi}} + 1 = [\varphi^2(t,y)]^{\frac{p}{p-1}}$$

Then, optimality conditions (i)-(ii) now turn into

$$\varphi_t^1 = -(1-p)\pi_t \quad \text{and} \quad \varphi^2(t,y) = \frac{1}{[1 + \hat{\pi}_t \cdot (e^y - \underline{1})]^{1-p}}.$$

Power utility $U_1(t,x) = U_2(x) = \frac{x^p}{p}$ with $p < 1, p \neq 0$

For each $t \in [0,T]$ define the map

$$q_t(\pi) := r_t\underline{1} - b_t - \frac{1}{2}\mathrm{diag}[a_t] + a_t(1-p)\pi_t - \int_{\mathbb{R}^d} \left[\frac{1}{[1+\pi\cdot(e^y - \underline{1})]^{1-p}}(e^y - \underline{1}) - h(y) \right] \nu_t(dy), \quad \pi \in K.$$

Theorem Let $\hat{\pi} = (\hat{\pi}_t)_{t\in[0,T]}$ be a (non-random) K-valued portfolio process satisfying

- $1 + \hat{\pi}_t \cdot (e^y - \underline{1}) > 0$ for all $(t,y) \in [0,T] \times \mathbb{R}^d$,
- The (non-random) process $\{q_t(\hat{\pi}_t),\ t \in [0,T]\}$ belongs to $\mathscr{D}$ and satisfies

$$g(t,\hat{\pi}_t) - \hat{\pi}_t q_t(\hat{\pi}_t) = \tilde{g}_K\left(t, q_t(\pi_t)\right), \text{ for all } t \in [0,T].$$

Then the portfolio process $\hat{\pi}$ is optimal.

Differential rates with short-selling constraint

Penalty function: $g(t,\pi_t) = -(R_t - r_t)(\pi_t - 1)^+, \;\; K = [0, +\infty)$

$$\tilde{g}_K(t,\zeta) = \begin{cases} 0, & \zeta > 0 \\ -\zeta, & \zeta \in [-(R_t - r_t), 0] \\ +\infty, & \zeta < -(R_t - r_t) \end{cases}$$

Effective domain $\mathscr{N}_t = [-(R_t - r_t), +\infty)$.

Suppose further X is spectrally positive: $\mathrm{supp}(\nu_t) \subset [0,\infty)$. Define

$$k_t(\pi) := \mu_t - a_t(1-p)\pi + \int_{\mathbb{R}^d} \frac{e^y - 1}{[1+\pi(e^y-1)]^{1-p}}\,\nu_t(dy), \quad \pi \geq 0$$

with

$$\mu_t := b_t + \frac{a_t}{2} + \int_0^1 y\,\nu_t(dy)$$

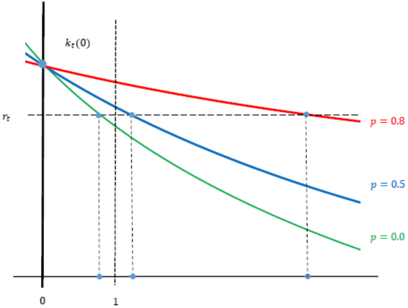
Then $q_t(\pi) = r_t - k_t(\pi) \in \mathscr{N}_t$ iff $k_t(\pi) \leq R_t$. Condition $g(t,\pi_t) - \pi_t q_t(\pi_t) = \tilde{g}_K(t, q_t(\pi_t))$ in this case reads

$$[r_t - k_t(\pi_t)]^- + \pi_t\,[r_t - k_t(\pi_t)] + (R_t - r_t)(\pi_t - 1)^+ = 0.$$

This can be solved for π_t using the inverse of k_t, if $a_t > 0$ or $R_t > \mu_t$.

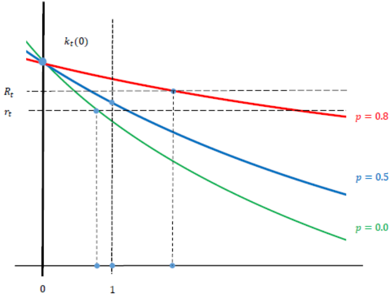
Equal rates: $R_t = r_t$

$$\hat{\pi}_t = k_t^{-1}(r_t)$$



Different rates: $R_t > r_t$

$$\hat{\pi}_t = \begin{cases} 0, & k_t(0) < r_t \\ k_t^{-1}(r_t), & k_t(1) \le r_t < k_t(0) \\ 1, & r_t < k_t(1) \le R_t \\ k_t^{-1}(R_t), & R_t < k_t(1) \end{cases}$$



THANK YOU!!