Chain Ladder and Granular Reserving

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This is partly joint work with
María D. (Lola) Martínez Miranda, Enno Mammen and Jens P. Nielsen

We want to predict the best estimate of the reserve within a stochastic model.

Stochastic chain ladder models
Kremer (1982) Verrall (1991) Mack (1993) Renshaw & Verall (1998) Wüthrich, Merz & Bühlmann (2008) Kuang, Nielsen & Nielsen (2009)

Major flaw:

- No modelling/discussion of the data generating process (individual data)

Chain ladder models with assumptions on individual level

Martínez-Miranda, Nielsen & Verrall (2012): Continuous Chain Ladder: Reformulating and generalising a classical insurance problem

Schiegl (2015): A model study about the applicability of the Chain Ladder method

Granular methods in discrete time

Drieskens et al. (2012)

Rosenlund (2012)

Pigeon, Antonio & Denuit (2013)

Godecharle & Antonio (2015)

Granular methods in continuous time (not related to chain ladder)

Arjas (1989)

Norberg (1993)

Antonio Plat (2014)

Zhao, Zhou & Wang (2009)

Zhao & Zhou (2010)

Continuous chain ladder: Granular method in continuous time

Martínez-Miranda et al. (2013)

The Chain Ladder Method:

Cumulative claims loss settlements		Development year 0	1	2	3	4	5	6	7
Claims occurrence year	2005	1232	2178	2698	3420	3736	3901	3949	3963
	2006	1469	2670	3378	4223	4684	4919	4975	
	2007	1652	3068	4027	4981	5586	5873		
	2008	1831	3465	4589	5676	6401			
	2009	2074	3993	5323	6563				
	2010	2434	4697	6358					
	2011	2810	4918						
	2012	3072							
CLM estimator for claims loss settlement factor			1,8508	1,3140	1,2422	1,1151	1,0491	1,0118	1,0035

3736+4684+5586+6401 = 20407

3420+4223+4981+5676 = 18300

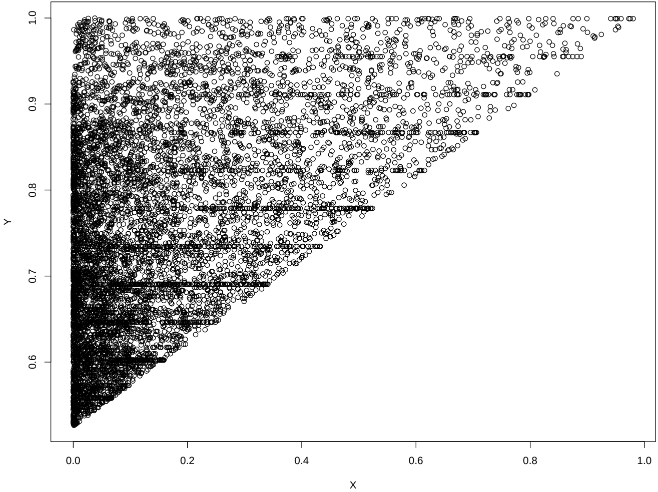
20407/18300 = 1,1151

Figure: Source: Weindorfer (2012)

The model

In-sample forecasting

$Y = \text{accident date},\ X = \text{development delay} \Rightarrow X + Y = \text{calendar time}$



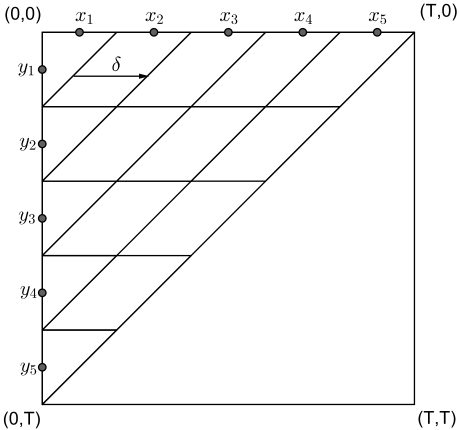


Figure: The usual aggregation of a triangle in the chain ladder method. The bin-width δ represents the length of a period.

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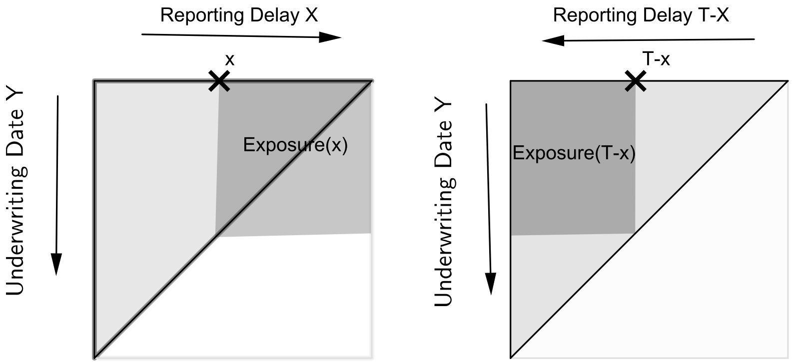

Figure: The exposure in forward moving time (left) and reversed time (right). Only in reversed time we observe the exposure.

We define the time reversed counting processes as

$$N_i(t) = I\left(T - X_i \leq t\right), \quad (i = 1, \ldots, n).$$

For the intensity we get almost surely

$$\begin{aligned} \nu_i(t) &= \lim_{h \downarrow 0} h^{-1} E\left[N_i\left\{(t+h)-\right\} - N_i(t-) \middle| \mathcal{F}^i_{t-}\right] \\ &= \alpha(t) Z_i(t). \end{aligned}$$

The Results

- the development factors are a discretisation of a hazard running in reversed development time

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- It is possible to to relax the assumption of independent row and column effects, without any parametric assumption.

In the following we consider two cases:

- The chain ladder case: X and Y are independent, i.e., $f(x.y) = f_1(x)f_2(y)$
- An extension: X and Y are allowed to be dependent, and $\alpha_2(y)\alpha_1(x)$

Central limit theorem for the density function in the case when X and Y are independent.

Theorem

Under standard regularity assumptions, for $t \in (0, T)$,

$$(nh)^{1/2}\left\{\widehat{f}_{l,h,K}(t) - f_l(t) - B_l(t)\right\} \to N\left\{0, \sigma_l^2(t)\right\} \quad (l = 1, 2), \quad n \to \infty,$$

in distribution, where
$B_l(t) = \frac{1}{2}\mu_2(\overline{K}^*) f_l^{''}(t) h^2 + o(h^2)$,
$\sigma_l^2(t) = \lim_{n\to\infty} nh \left\langle V_l \right\rangle_t = R(\overline{K}^*) f_l(t) F_l(t) \gamma_l(t)^{-1}$,
$\gamma_l(t) = \mathrm{pr}(Z_l^1(t) = 1)$.

Central limit theorem for a family of data-driven bandwidth selectors for the case X and Y being independent.

Theorem

Under (T1)–(T4), the bandwidth selector $\widehat{h}_{\mathrm{ICV}}$ of the local linear survival density estimator in the original time direction satisfies

$$n^{3/10}\left(\widehat{h}_{\mathrm{ICV}} - h_{\mathrm{MISE}}\right) \to N\left(0, \sigma_1^2\right), \quad n^{3/10}\left(\widehat{h}_{\mathrm{ICV}} - h_{\mathrm{ISE}}\right) \to N\left(0, \sigma_2^2\right),$$

where

$$\sigma_1^2 = S_1 \int \left\{ \sum_{j=1}^{J} m_j \frac{R(K)}{R(\bar{L}_j)} (H_{L_j} - G_{L_j})(\rho_j u) \right\}^2 du,$$

$$\sigma_2^2 = S_1 \int \left\{ \sum_{j=1}^{J} m_j \frac{R(K)}{R(\bar{L}_j)} (H_{L_j} - G_{L_j})(\rho_j u) - H_K(u) \right\}^2 du + S_2,$$

$$S_1 = \frac{2}{25}\frac{\int S^2(t)f^2(t)\widetilde{w}^2(t)\mathrm{d}t}{R^{7/5}(K)\mu_2^{6/5}(K)\left\{\int f''(t)^2\gamma(t)\widetilde{w}(t)\ dt\right\}^{3/5}\left\{\int f(t)S(t)\widetilde{w}(t)\ dt\right\}^{7/5}},$$

$$S_2 = \frac{4}{25}\frac{\int f''(t)^2 S(t)f(t)\widetilde{w}^2(t)\gamma(t)\mathrm{d}t - \int\left\{\int_t^T f''(u)f(u)\widetilde{w}(u)\gamma(u)\mathrm{d}u\right\}^2\alpha(t)\gamma^{-1}(t)\mathrm{d}}{R^{2/5}(K)\mu_2^{6/5}(K)\left\{\int f(t)S(t)\widetilde{w}(t)\mathrm{d}t\right\}^{2/5}\left\{\int f''(t)^2\gamma(t)\widetilde{w}(t)\mathrm{d}t\right\}^{8/5}}$$

and $G_K(u) = I(u \neq 0)\left\{\overline{K}^{**}(u) - \overline{K}^{**}(-u)\right\}$, and
$H_K(u) = I(u \neq 0)\int \overline{K}^{*}(v)\left\{\overline{K}^{**}(u+v) - \overline{K}^{**}(-u+v)\right\}\mathrm{d}v$, with

$$\overline{K}^{**}(u) = -\frac{\mu_2(K) - \mu_1(K)u}{\mu_2(K) - \{\mu_1(K)\}^2}\{K(u) + uK'(u)\} + \frac{\mu_1(K)u}{\mu_2(K) - \{\mu_1(K)\}^2}K(u).$$

Central limit theorem for the estimation error of the components of a d-dimensional hazard $\alpha(t|Z) = \alpha_0(t) \times \alpha_1(z_1) \times \cdots \times \alpha_d(z_d)$.

Theorem

With probability tending to one there exists a solution $\widehat{\delta} = (\widehat{\delta}_0, \ldots, \widehat{\delta}_d)$ *of the equation* $\widehat{\mathcal{F}}(f_0, \ldots, f_d) = 0$ *with*

$$||\widehat{\delta} - \overline{\delta}||_{p,\infty} = o_p(n^{-2/5}).$$

For this solution we get that

$$n^{2/5}\{(\widehat{\alpha}_j - \alpha_j)(x_j) - \alpha_j(x_j)[(I-\pi)^{-1}(\widehat{\mu}^B - \overline{\delta}^{B,*})]_j(x_j)\} \to N(0, \alpha_j^2(x_j)\sigma_j^2(x_j)),$$

in distribution, for x_j *(*$0 \leq j \leq d$*) with* $p_j(x_j) > 0$.

Some finite sample results

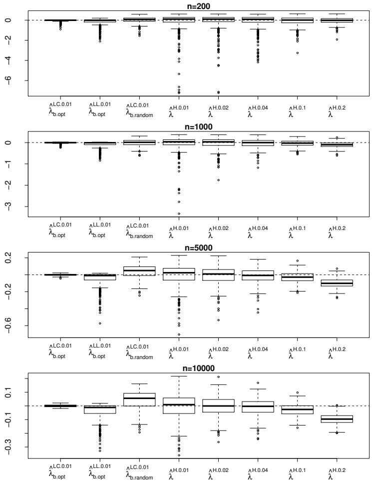
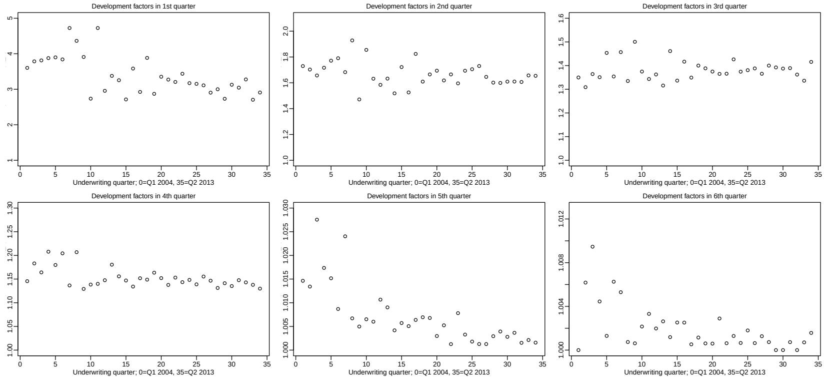


Figure: Boxplot results over 500 repetitions for the relative estimation error of the reserve. The development delay, X, has a Beta distribution with parameters $(2, 5)$, and the underwriting date density, Y, is linear increasing, $f_2(y) = 2y$. For $n = 200, 1000 :$ b.random $\in [0.05, 0.3]$, for $n = 1000, 5000 : b.random \in [0.05, 0.25]$

Data Application

- Data: Number of claims reported between 2004 and 2013.
- $n = 58180$ claims (sample size).
- The triangular data is given as $\{(X_1, Y_1), \ldots, (X_n, Y_n)\}$.
  - X_i = underwriting date of claim i.
  - Y_i = reporting delay (i.e. the time between underwriting date and the date of report of a claim) in days.

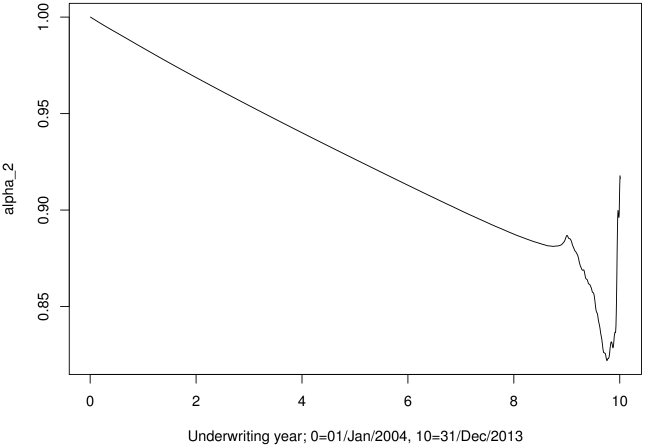
Figure: Development factors of the first six quarter for individual underwriting quarter.



We assume a multiplicative structure:

$$\alpha(y, x) = \alpha_2(y)\alpha_1(x)$$

Figure: Estimated correction factor $\alpha_2(y)$ in the real data application.



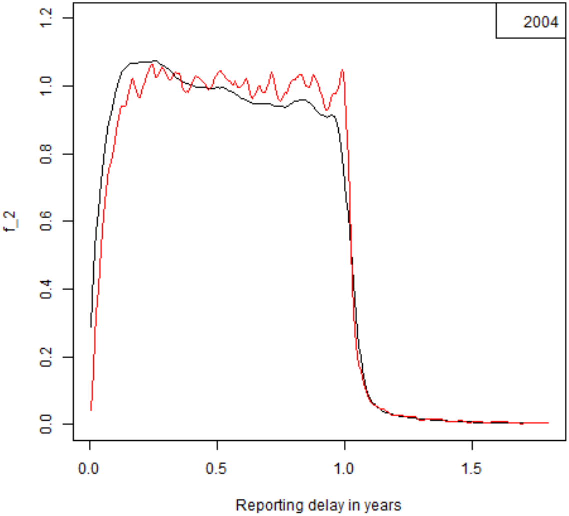

2004
f_2
0.0
0.2
0.4
0.6
0.8
1.0
1.2
0.0
0.5
1.0
1.5
Reporting delay in years

Table: Number of claims forecasts in the real data application. In quarters; $1 = 2014\ Q1,\ 39 = 2022\ Q3$.

Future quarter	1	2	3	4	5	6	7	8	9	10	11	12 – 39	Tot.
CV	1027	733	465	201	15	5	3	2	1	1	1	0	2452
DO	970	684	422	166	14	5	3	2	1	1	1	0	2270
CLM	948	651	387	148	12	5	3	2	1	1	1	0	2160
MH	872	621	400	130	53	7	4	3	2	1	1	1	2193

Papers can be found on my webpage:
https://sites.google.com/site/munirhiabu/publications

Papers under review or revision

1. **In-sample forecasting of local linear survival densities** (with Mammen, E., Martinez-Miranda, M. D. and Nielsen, J. P.)
   Submitted to *Biometrika*
   [preprint] [Supplementary material]

2. **On the relationship between classical chain ladder and granular reserving**
   Submitted to *Scandinavian Actuarial Journal*
   [preprint]

3. **Continuous chain-ladder with paid data: the theoretical foundation**
   Submitted to *Statistics & Probability Letters*
   [preprint]

Work in progress

1. **Smooth backfitting of multiplicative structured hazards** (with Mammen, E., Martinez-Miranda, M. D. and Nielsen, J. P.)