

On de Finetti's problem under a time of ruin constraint

Spectrally negative reserves with completely monotone Lévy
measure

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Outline

De Finetti's Problem

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Exponential claims case

Spectrally Negative Lévy reserves

Numerical example

Extensions

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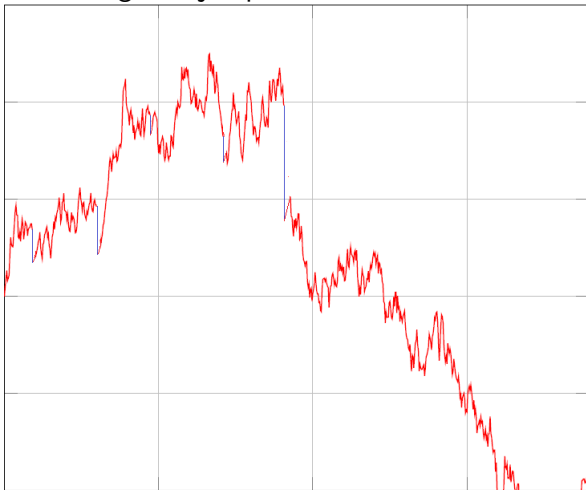
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Adapted, non-negative, non-decreasing, *càglàd* processes with $D_0 = 0$.
- ▶ Surplus Process, $X^D := X - D$:
Process D cannot lead to ruin, so $D_{t+} - D_t \leq X_t^D \vee 0$.
Let Θ be the set of such strategies.

Objective Function: The company's objective is to maximize

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De Finetti's problem [De Finetti, 1957] consists in finding

$$\sup_{D \in \Theta} \mathcal{V}^D(x_0).$$

The solution to this problem has been extensively studied for different classes of reserve processes ([Schmidli, 2008], [Loeffen, 2008].)

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Motivation

There exists a trade-off between stability and profitability. The alternative problem of minimizing ruin probability means no dividend payment and profits are 0.

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There exists a trade-off between stability and profitability. The alternative problem of minimizing ruin probability means no dividend payment and profits are 0.

Some problems have been proposed to deal with this trade-off:
[Paulsen, 2003],[Thonhauser and Albrecher, 2007],
[Loeffen and Renaud, 2010].

The Problem

Inspired by previous work of [Thonhauser and Albrecher, 2007] we introduce a constraint on the time of ruin and state the following problem:

$$V(x) := \sup_{D \in \Theta} \mathcal{V}^D(x) \quad (\text{P})$$

$$\text{s.t. } \mathbb{E}_x \left[\int_0^{\tau^D} e^{-qs} ds \right] \geq K,$$

for $K \leq \frac{1}{q}$.

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$$\text{s.t. } \mathbb{E}_x \left[\int_0^{\tau^D} e^{-qs} ds \right] \geq K,$$

for $K \leq \frac{1}{q}$. Possibly:

- ▶ $K = K_T := \int_0^T e^{-qs} ds$ for some $T > 0$.
- ▶ $\mathbb{E}_x[e^{-q\tau^D}] \leq \mathbb{P}_x(\tau^D < \infty) \leq \epsilon$.

In order to solve this problem we use the Lagrangian approach to reformulate our problem. We first define the following function

$$\mathcal{V}_\Lambda^D(x) := \mathbb{E}_x \left[\int_0^{\tau^D} e^{-qs} dD_s + \Lambda \int_0^{\tau^D} e^{-qs} ds \right] - \Lambda K \quad (\text{L})$$

for $\Lambda \geq 0$.

Then, (P) is equivalent to $\sup_{D \in \Theta} \inf_{\Lambda \geq 0} \mathcal{V}_\Lambda^D(x)$ since

$$\inf_{\Lambda \geq 0} \mathcal{V}_\Lambda^D(x) = \begin{cases} \mathcal{V}^D(x) & \text{if } \mathbb{E}_x \left[\int_0^{\tau^D} e^{-qs} ds \right] \geq K \\ -\infty & \text{otherwise .} \end{cases}$$

Dual problem

The dual problem of (P) is defined as

$$\inf_{\Lambda \geq 0} \sup_{D \in \Theta} \mathcal{V}_{\Lambda}^D(x). \quad (\text{D})$$

Remember that $(\text{P}) \leq (\text{D})$ always holds.

Therefore, once (D) is solved, the main goal is to prove that there is no duality gap, that is

$$\sup_{D \in \Theta} \inf_{\Lambda \geq 0} \mathcal{V}_{\Lambda}^D(x) = \inf_{\Lambda \geq 0} \sup_{D \in \Theta} \mathcal{V}_{\Lambda}^D(x).$$

Recall

$$\mathcal{V}_\Lambda^D(x) = \mathbb{E}_x \left[\int_0^{\tau^D} e^{-qt} dD_t + \Lambda \int_0^{\tau^D} e^{-qt} dt \right] - \Lambda K.$$

To solve (D), we can use previous results on de Finetti's problem and compute

$$V_\Lambda(x) := \sup_{D \in \Theta} \mathcal{V}_\Lambda^D(x) \quad (\text{Pa})$$

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Where to start? A simple scenario.

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Assume the reserves process follow the Cramér-Lundberg model with exponentially distributed claim sizes.

Theorem

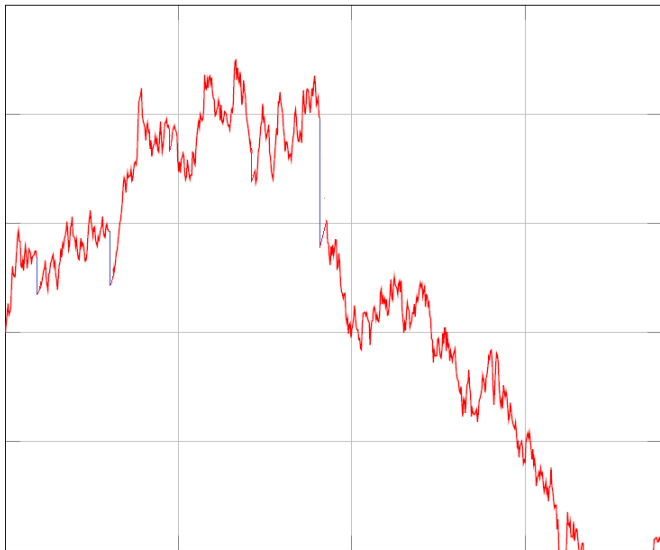
Let $x \geq 0$, $K \geq 0$ and $V(x)$, $V_\Lambda(x)$ be the optimal solutions to (P), (Pa), respectively, with initial value x . Then, strong duality holds, i.e.,

$$\inf_{\Lambda \geq 0} V_\Lambda(x) = V(x).$$

- ▶ Key point: **Barrier strategies are optimal.**

[Hernández and J., 2015]

Can we do more?



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Some definitions

- ▶ X is a *spectrally negative Lévy process*, SNLP, if its Lévy triple (a, σ, Π) is such that Π is supported on $(-\infty, 0)$, i.e., they have no positive jumps, and do not have monotone paths a.s.

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- ▶ For SNLP we define the *Laplace exponent* and its inverse by

$$\psi(\theta) := \log(\mathbb{E}[e^{\theta X_1}]) \quad \theta \geq 0,$$

$$\Phi(q) = \sup\{\theta \geq 0 : \psi(\theta) = q\}, \quad q \geq 0.$$

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- ▶ There exist a family of functions called q -scale $W^{(q)} : \mathbb{R} \rightarrow [0, \infty)$ defined for each $q \geq 0$ s.t. $W^{(q)}(x) = 0$ for $x < 0$ and $W^{(q)}$ is a strictly increasing and continuous function on $[0, \infty)$ whose Laplace transform satisfies,

$$\int_0^\infty e^{-\beta x} W^{(q)}(x) dx = \frac{1}{\psi(\beta) - q} \quad \text{for } \beta > \Phi(q). \quad (1)$$

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- ▶ For each $q \geq 0$ define $Z^{(q)}(x) := 1 + q \int_0^x W^{(q)}(z) dz$, for $x \in \mathbb{R}$.

Consider a barrier strategy at level b , i.e., $D_t^b := (b \vee \bar{X}_t) - b$, $t \geq 0$, so $X_t^{D^b} = b - [(b \vee \bar{X}_t) - X_t]$. This motivates the following: Given a Lévy process Y we define

$$\hat{Y}_t^s := (s \vee \bar{Y}_t) - Y_t, \quad t \geq 0. \quad (2)$$

For such processes define also the exit time

$$\hat{\sigma}_k^s := \inf\{t > 0 : \hat{Y}_t^s > k\}. \quad (3)$$

Therefore,

$$X_t^{D^b} = b - \hat{X}_t^b$$

and

$$\tau^b := \tau^{D^b} = \hat{\sigma}_b^b.$$

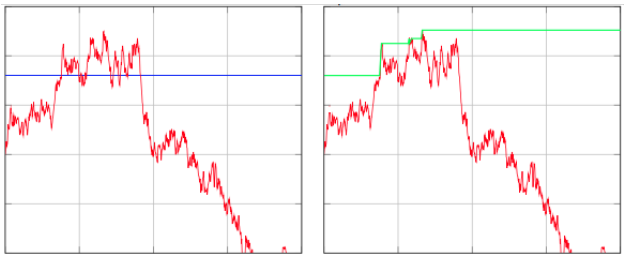


Figure: Process X_t and process $(b \vee \bar{X}_t)$ (green line)

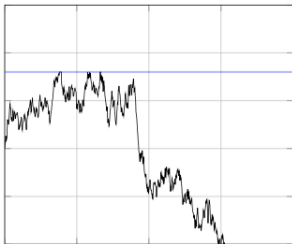


Figure: $X_t^{D^b} = b - \hat{X}_t^b$

The following results are the link between (Pa) and the q -scale functions:

Theorem ([Gerber, 1972])

Let $b > 0$ and consider the process D^b as before. For $x \in [0, b]$,

$$\mathbb{E}_x \left[\int_0^{\tau^b} e^{-qs} dD_s^b \right] = \frac{W^{(q)}(x)}{W^{(q)'(b+)}, \quad (4)$$

where $W^{(q)'(b+)}$ is understood as the right derivative of $W^{(q)}$ at b .

Theorem ([Avram et al., 2004])

Let $b > 0$ and consider the process D^b as before. For $x \in [0, b]$,

$$\mathbb{E}_x \left[e^{-q\tau^b} \right] = Z^{(q)}(x) - q \frac{W^{(q)}(x)}{W^{(q)'}(b)} W^{(q)}(b).$$

Therefore,

$$\mathbb{E}_x \left[\int_0^{\tau^b} e^{-qs} ds \right] = \frac{W^{(q)}(x)}{W^{(q)'}(b)} W^{(q)}(b) - \int_0^x W^{(q)}(z) dz. \quad (5)$$

(L) for a barrier strategy

Expressions (5) and (4) yields the following result.

Proposition ([Loeffen, 2009])

For a sufficiently smooth q -scale function $W^{(q)}$, the value of (L) for the barrier strategy at level $b \geq 0$ is given by

$$\mathcal{V}_{\Lambda}^{D^b}(x) = \begin{cases} W^{(q)}(x) \left[\frac{1 + \Lambda W^{(q)}(b)}{W^{(q)'}(b)} \right] - \Lambda \int_0^x W^{(q)}(z) dz - \Lambda K & \text{if } x \leq b \\ x - b + \mathcal{V}_{\Lambda}^{D^b}(b) & \text{if } x > b. \end{cases} \quad (6)$$

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Let $\zeta_\Lambda : [0, \infty) \rightarrow \mathbb{R}$ be defined by

$$\zeta_\Lambda(x) := \frac{1 + \Lambda W^{(q)}(x)}{W^{(q)'}(x)}, \quad x > 0 \quad (7)$$

and $\zeta_\Lambda(0) := \lim_{x \downarrow 0} \zeta_\Lambda(x)$. In [Loeffen, 2009] it is shown that when the Lévy measure Π has completely monotone density, this function has a unique maximum.

Definition

A function f is said to be *completely monotone* if $f \in C[0, \infty)$, $f \in C^\infty(0, \infty)$ and satisfies $(-1)^n \frac{d^n}{dx^n} f(x) \geq 0$

Value Function of (Pa)

Theorem ([Loeffen, 2009])

Suppose the Lévy measure Π of the spectrally negative Lévy process X has a completely monotone density. Then, for each $\Lambda \geq 0$, the optimal strategy of the problem

$$V_{\Lambda}(x) = \sup_{D \in \Theta} \mathbb{E}_x \left[\int_0^{\tau^D} e^{-qt} dD_t + \Lambda \int_0^{\tau^D} e^{-qt} dt \right] - \Lambda K. \quad (\text{Pa})$$

consists of a barrier strategy at level b_{Λ}^ given by the maximum of $\zeta_{\Lambda}(x)$.*

We also have the converse of the previous theorem:

Proposition

For each $b \in (b_0^, \infty)$ there exists a Λ such that the barrier strategy at level b , is optimal for (Pa) with Λ .*

Theorem

Let $x \geq 0$, $K \geq 0$, and $V(x)$, $V_\Lambda(x)$ be the optimal solution to (P), (Pa), respectively, with initial value x , then, strong duality holds, i.e.,

$$\inf_{\Lambda \geq 0} V_\Lambda(x) = V(x).$$

Proof. Let τ^b be the time of ruin of the barrier strategy with level b and initial value $x \in [0, b]$. Let

$$\Psi_b(x) = \mathbb{E}_x \left[\int_0^{\tau^b} e^{-qs} ds \right].$$

To obtain the optimal value function of (P) we need to show (b^*, Λ^*) that certify strong duality. To do so we distinguish the following 4 cases depending on the initial value x and the parameter K :

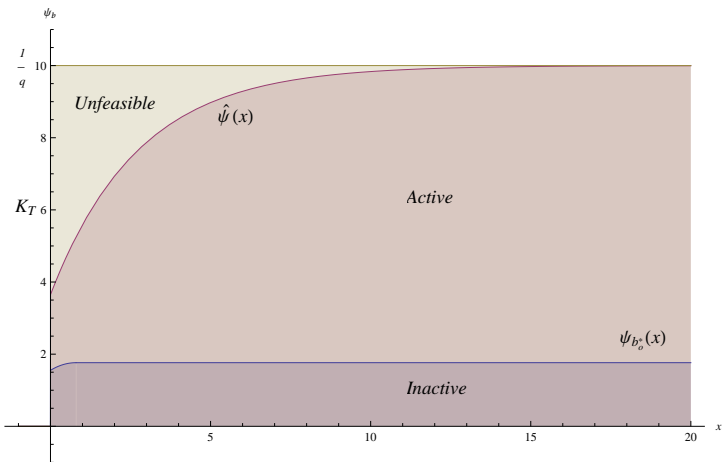


Figure: Characterization of the solution of (P).

Numerical Example

We illustrate the previous result for X with Lévy triple $\left(10, 0.1, \frac{dx}{|x|^{1.8}} 1_{(-\infty, 0)}\right)$, that is an 0.8–stable process with diffusion.

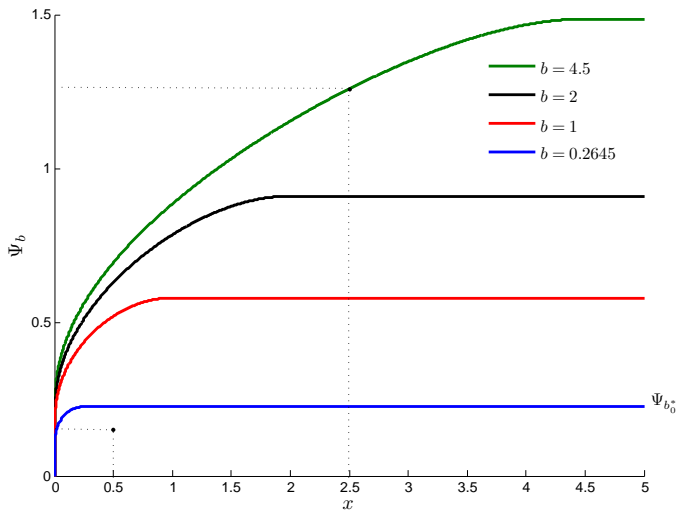


Figure: $\Psi_b(x)$ for different barrier levels. Inactive and bidding constraint cases.

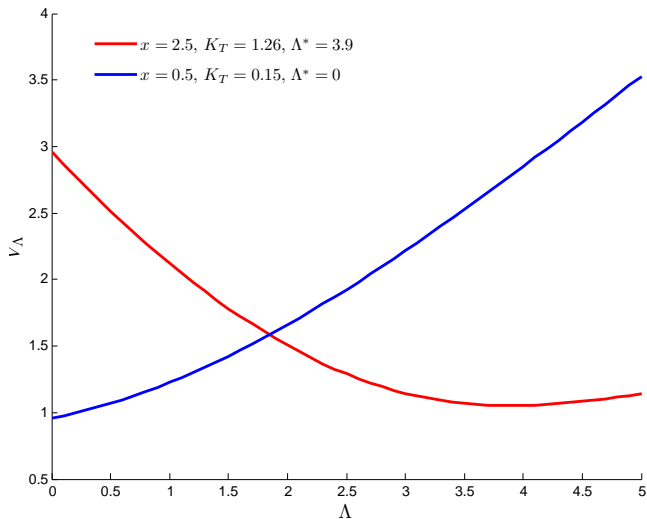


Figure: $V_\Lambda(x)$ for different initial values. Inactive and bidding constraint cases.

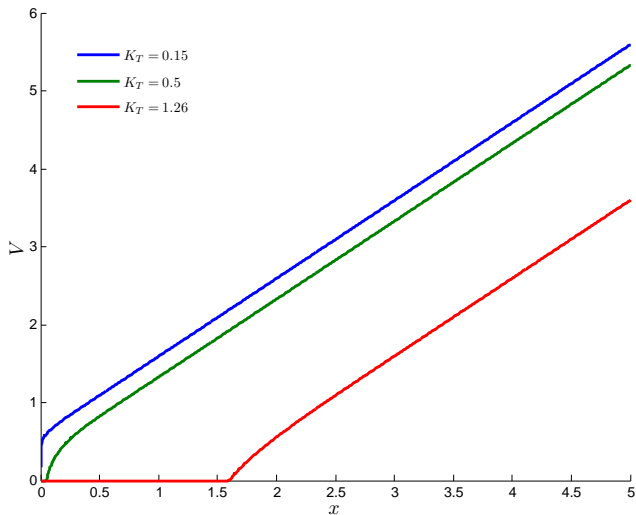


Figure: Value functions for constrained and unconstrained problems for different values of K_T .

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- ▶ Dual model (spectrally positive Levy Processes). ✓
- ▶ Optimal dividend with fixed transaction cost. (Impulse control and (S, s) strategy).
- ▶ Study other types of restrictions such that:

$$\mathbb{P}(\tau^D \leq T | X_0^D = x_0) \leq \epsilon.$$

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Thank You