Heterogeneous Archimedean copulae and t-copulae in credit portfolio modeling

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Topics

Topics

Basics

A copula C is a multivariate distribution function with uniform distributed marginals,

$$c : [0,1]^d \to [0,1] \text{ s.t. } c(1,..,1,x,1,..,1) = x \text{ and}$$
$$C(x_1,..,x_d) \text{ is increasing in each component}$$
$$\forall a_1 \leq b_1, ..a_d \leq b_d, a_i, b_j \in [0,1]$$
$$\sum_{i_1=1,2} \cdots \sum_{i_d=1,2} (-1)^{i_1+\cdots+i_d} C(x_{1i_1},..,x_{di_d}) \geq 0$$
$$\text{where } u_{j1} = aj, u_{j2} = b_j, j = 1,..,d$$

An easy way to generate copulae is from a given multivariate distribution function $F : (-\infty,\infty)^d \to [0,1]$ by

$$u(x) = u_F(x) = F(F_1^{-1}(x_1),..,F_d^{-1}(x_d)).$$

Gaussian Copula

As an example the classical normal copula with correlation matrix Σ is obtained by

$$\begin{aligned} u_{N,\Sigma}(x) &= N(N^{-1}(x_1), .., N^{-1}(x_d); \Sigma) \qquad (1) \\ &= \frac{1}{\sqrt{(2\pi)^d \det(\Sigma)}} \int_{-\infty}^{N^{-1}(x_1)} \cdots \int_{-\infty}^{N^{-1}(x_d)} \exp(-\frac{1}{2} y`\Sigma y) dy_1 \cdots dy_d \end{aligned}$$

Because of lack of tail/dependency, normal copula is not adequat for modeling the dependency structure of financial time series.

Non-Gaussian Copula

Other copula have a taii-dependency, e.g. the t-copula, the copula of the multivariate t-distribution, which can be generated as

$$X = WZ$$

where $N(0, \Sigma)$ distribution and

$$W = \sqrt{\frac{\nu}{\chi_\nu^{-1}(S)}} \tag{2}$$

with S uniform and χ_ν^{-1} is the inverse of the distribution function of a χ^2-distributed random variable with $\nu > 0$ degrees of freedom (dof).

$$\begin{aligned} u_{t,\nu,\Sigma} &= P[t_\nu(X_1) \leq x_1, .., t_\nu(X_d) \leq x_d] \\ &= t_{\nu,\Sigma}(t_\nu^{-1}(x_1), .., t_\nu^{-1}(x_d)) \end{aligned} \tag{3}$$

where t_ν is the univariate t-distribution function and $t_{\nu,\Sigma}$ the multivariate. From a risk management point of view, one can W^{-1} interprete as a non-linear factor which has an impact on all components of X. Especially if $\chi_\nu(S)$ is small, there is a large impact on all factors.

Topics

Heteregeneity

- t-copula has only one additional parameter, differences in pairwise tail-dependency are still only driven by correlation matrix.
- In order to obtain heterogeneous tail-dependency beyond correlation one has to introduce t-copula/distribution with multiple parameters. Most of them, like nested, grouped, or hierachical require a certain structure.
- We will follow Luo and Shevchenko (2007), which requires no structure .

Literatur:

Okhrin, O., Okhrin, Y. , Schmid, W. (2013). Determining the structure and estimation of hierarchical Archimedean copulas

A. McNeil (2008). Sampling nested Archimedean copulas.

Daul, S., E. De Giorgi, F. Lindskog and A. McNeil (2003). The grouped t-copula with an application to credit risk.

Luo and Shevchenko (2007), The t copula with Multiple Parameters of Degrees of Freedom: Bivariate Characteristics and Application to Risk Management

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Heterogenous t-copula

Instead of multiplyiing a normaly distributed vector Z by one single W we multiply each component in (2) by a specific W_i

$$W_i = \sqrt{\frac{\nu_i}{\chi^{-1}_{\nu_i}(S)}} \tag{4}$$

keeping the same uniform distributed S for all components. This gives the heterogeneous t-distribution

$$t_{\nu_1,..,\nu_d,\Sigma}$$

as the distribution of the vector $Y = (Y_1, .., Y_d)^T$ with

$$Y_i = W_i \cdot Z_i \tag{5}$$

and the corresponding copula

$$\begin{aligned} u_{t,\nu_1,..\nu_d,\Sigma} &= P\left[t_{\nu_1}(X_1) \leq x_1, .., t_{\nu_d}(X_d) \leq x_d\right] \\ &= t_{\nu_1,..\nu_d,\Sigma}\left(t^{-1}_{\nu_1}(x_1), .., t^{-1}_{\nu_d}(x_d)\right) \end{aligned} \tag{6}$$

Topics

Archimedean Copula

Another class of Copulae are the Archimedean. They are defined in terms of a generator function ϕ. This is a decreasing function from $[0,1]$ to $[0,\infty]$ with $\phi(0)=\infty$ and $\phi(1)=0$. Then

$$C(x_1,..,x_d)=\phi^{-1}(\phi(x_1)+\cdots\phi(x_d)) \tag{7}$$

is a copula. Since this is an exchangeable copula on dimension $d\geq 2$ it assumes a homogenous dependency between all marginal distribution. A generic way to incorporate heteregeneity of dependencies we introduce a straight forward procedure, which is based on the simulation scheme for Archimedian copulae.

One particular class of functions satisfying the properties requested for generator ϕ are those derived from distributions on $[0, \infty)$. Namely if $\phi = \hat{G}^{-1}$ where $\hat{G}$ is the Laplace-transform of a distribution function G, i.e.

$$\begin{aligned} \hat{G}(s) &= \int_0^\infty e^{-sy} G(dy) \\ &= E[e^{-sX}] \end{aligned}$$

if X is distributed according to G. This gives the following results

Theorem (cf. Mc Neil et al, ...)

Let X be distributed according to G and $X_1, ..., X_d$ i.i.d uniform. Then

$$U = (\hat{G}(-\ln(X_1)/X, .., \hat{G}(-\ln(X_d)/X)$$

is the archimedian copula with generator $\hat{G}^{-1}$.

The corresponding algorithm to generate a sample with Archimediamn Copula is therefore:

1. Generate a sample x_0 according to G
2. Generate independent sample $x_1, .., x_d$ from the uniform distribution
3. $u = (\hat{G}(-\ln(x_1)/x_0), ..(\hat{G}(-\ln(x_d)/x_0))$

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Heterogeneous Archimedean copula

Once we are at this state we can easily define the heterogeneous Archimedian copula. Let $G_1, ..G_d$ be d-distribution functions on $[0, \infty)$.

Theorem

Let $X, X_1, .., X_d$ independent uniform on $[0, 1]$. Then

$$U = (\hat{G}_1[-\ln(X_1)/G_1^{-1}(X)], .., \hat{G}_d[-\ln(X_d)/G_d^{-1}(X)])$$

is a copula with distribution function

$$C(u_1, .., u_d) = \int_0^1 e^{-\sum_{j=1}^d \hat{G}_j(u_j)G_j^{-1}(u)} du \tag{8}$$

Topics

Capital Allocation

How do you allocate the risk, e.g. Value-at-Risk or Expected shortfall of a portfolio X

$$X = \sum_{i=1}^{n} X_i$$

to the subportolios X_i?

Generally if $\rho(X)$ is the risk of the portfolio, then the gradient allocation

$$\Lambda(Y;X) \quad = \quad \nabla_Y \rho(X) = \lim_{\epsilon \to 0} \frac{\rho(X+\epsilon Y) - \rho(X)}{\epsilon} \tag{9}$$

is the standard method for allocation. If ρ is 1-homegeneous, it has the full allocation property which follows from the Euler property

$$\rho(X) \quad = \quad \sum_{i=1}^{n} \nabla_{X_i} \rho(X) \tag{10}$$

Expected Shortfall Allocation

If $\rho(X) = E[X|X \geq q_\alpha(X)]$ which equals in regular cases the coherent average value at risk

$$\int_\alpha^1 q_\beta(X)d\beta$$

the gradient equals

$$\Lambda(Y;X) \quad = \quad E[Y|X \geq q_\alpha(X)] \tag{11}$$

Implementation and Example

- Portfolio
  IACPM test portfolio 2006. Net-Exposure around 38,000,000,000 USD. 3000 Obligors. Exposure from 600,000,000 to 200,000. 7 non-defaulted rating classes, with PD from 0.0001 to 0.225. More than 1/3 in the middle rating class and around 3% in the best and worst rating class.
- Factor modell
  7 regions. 17 industries. Factor building based on asset-value time-series. Factor weights of obligors based on variance decomposition. Estimation of a 24-dimensional heterogeneous t-copula. Error assumed to be normal.
- Parameters
  - R-squared of variance decomposition in the range between 5.8% and 48.5%
  - Heterogeneous degree of freedom parameters varied between 3.1 and 8. Smallest dof with non-banking financial industry and largest with Germany

Results

- The overall expected shortfall increased by 12% at the 99.9% VaR
- The overall VaR by 9%
- The largest change in allocated capital (ES(99.7%) was 233%, from 0.1341% to 0.4468%, and -25% from 0.1631% to 0.1223%.
- The order of best to worst customer changed considerable. Largest improvement by 231 ranks from 4% to 3.9% and largest detoriation by 322 ranks from 3.3% to 4.8%.

Summary

- Tail-dependency must be incorporated into risk management
- Heterogeneity of tail-dependence is important for capital allocation
- Heterogeneous t-copula is a possible implementation to caputure these feature
- General heterogenous Archimdean copulas provide also a tool set for heterogenous tail-dependency

Thank you for your attention!

Literature

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