On the behavior of the price impact in the Kyle-Back model

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Main references

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The market model

We consider a market with two assets, a stock of a firm and a bank account with interest rate r equal to zero for the sake of simplicity. The trading is continuous in time over the period $[0, \infty)$ and it is order driven. There is a release time τ where the *fundamental* value of stock is revealed. The fundamental value process is denoted by V. We shall denote the market price of the stock at time t by P_t. Just after the revelation time the market price and the fundamental value will coincide.

The market model

There are three kinds of traders. A *large* number of liquidity traders, who trade for liquidity or hedging reasons, an informed trader or insider, who has privileged information about the firm and can deduce the fundamental price, and the market makers, who set the price and clear the market.

The market model

Let X be the demand process of the informed trader. At time t, her information is represented by the σ-field $\mathcal{H}_t$ and her flow of information is represented by the filtration $\mathbb{H} = (\mathcal{H}_t)_{t \geq 0}$. The informed trader, like any other trader, observes the market prices P and, in addition, she has access to some signal process η (càdlàg) related to the firm value, also she knows a random effect in the price impact ρ. Moreover, she will observe the random time τ.

- $\mathcal{H}_t = \sigma(P_s, \eta_s, \mathbf{1}_{\{\tau \leq s\}}, \rho_s, 0 \leq s \leq t)$.

The insider has access to the fundamental value and, in terms of the insider's information flow, this is assumed to be a $\mathbb{H}$-martingale

The market model

Let Z be the *aggregate* demand process of the liquidity traders. They are perceived as constituting noise in the market, thus also called *noise* traders. We assume that Z is a continuous $\mathbb{H}$-martingale, independent of η and V.

The market model

Market makers "clear" the market fixing the market prices. They rely on the information given by the total aggregate demand $Y := X + Z$ which they observe, the random factor affecting the price impact, ρ, and they observe the release time. Hence their information flow is: $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$, where $\mathcal{F}_t = \sigma(Y_s, \mathbf{1}_{\{\tau \leq s\}}, \rho_s, 0 \leq s \leq t)$.

Moreover we suppose that market prices follow a pricing rule of the form:

$$P_t = H(t, \xi_t), t \geq 0$$

involving

$$\xi_t := \int_0^t \lambda(s, \rho_s, P_s) \mathrm{d}Y_s,$$

where $\lambda \in C^{1,1,2}$ is a strictly positive function that we call it *price impact* (also named *market depth*), $H \in C^{1,2}$, and $H(t, \cdot)$ is strictly increasing for every $t \geq 0$. Hence we have the following definition.

Definition

Denote the class of such pairs (H, λ) above by $\mathfrak{H}$. An element of $\mathfrak{H}$ is called a pricing rule.

Due to the competition among market makers, the market prices are *rational*, or *competitive*, in the sense that

$$P_t = \mathbb{E}(V_t | \mathcal{F}_t), \quad t \geq 0.$$

The informed trader is assumed to be risk-neutral and she aims maximizing her final wealth. Let W be the wealth process corresponding to insider's portfolio X.

Definition

A strategy X is called *optimal* with respect to a price process P if it maximizes $\mathbb{E}(W|\mathcal{H}_0)$.

The equilibrium

Definition

Let $(H, \lambda) \in \mathfrak{H}$ and consider a strategy X. The triple (H, λ, X) is an equilibrium, if the price process $P_{\cdot} := H(\cdot, \xi_{\cdot})$ is rational, given X, and the strategy X is optimal, given (H, λ).

Insider's wealth

To illustrate the relationship among the processes V, P, X, and W we first consider a multi-period model where trades are made at times $i = 1, 2, \dots N$, and where $\tau = N$ is random. If at time $i-1$, there is an order of buying $X_i - X_{i-1}$ shares, its cost will be $P_i(X_i - X_{i-1})$, so, there is a change in the bank account given by $-P_i(X_i - X_{i-1})$.Then the total change is

$$-\sum_{i=1}^{N} P_i(X_i - X_{i-1}),$$

and due to the convergence of the market and the fundamental prices just after time $\tau = N$, there is the extra income: $X_N V_N$. So, the total wealth is

$$\begin{aligned} W_N &= -\sum_{i=1}^{N} P_i(X_i - X_{i-1}) + X_N V_N \\ &= -\sum_{i=1}^{N} P_{i-1}(X_i - X_{i-1}) - \sum_{i=1}^{N} (P_i - P_{i-1})(X_i - X_{i-1}) + X_N V_N \end{aligned}$$

Consider now the continuous time setting where we have the processes X, P, and V, and we take N trading periods, where N is random and the trading times are: $0 \leq t_1 \leq t_2 \leq \ldots \leq t_N = \tau$, then we have

$$W_\tau = -\sum_{i=1}^{N} P_{t_{i-1}}(X_{t_i} - X_{t_{i-1}}) - \sum_{i=1}^{N}(P_{t_i} - P_{t_{i-1}})(X_{t_i} - X_{t_{i-1}}) + X_N V_N,$$

so if the time between trades goes to zero we will have formally

$$\begin{aligned} W_\tau &= X_\tau V_\tau - \int_0^\tau P_{t-} \mathrm{d}X_t - [P, X]_\tau \\ &= \int_0^\tau X_{t-} \mathrm{d}V_t + \int_0^\tau V_{t-} \mathrm{d}X_t + [V, X]_\tau - \int_0^\tau P_{t-} \mathrm{d}X_t - [P, X]_\tau \\ &= \int_0^\tau (V_{t-} - P_{t-})\, \mathrm{d}X_t + \int_0^\tau X_{t-} \mathrm{d}V_t + [V, X]_\tau - [P, X]_\tau \end{aligned}$$

Insider's optimal strategy

Insider's strategy X is optimal if it maximizes

$$\begin{aligned} & \mathbb{E}\left(W_\tau \middle| \mathcal{H}_0 \right) \\ = \; & \mathbb{E}\left(\int_0^\tau (V_t - H(t, \xi_{t-}))\mathrm{d}X_t + \int_0^\tau X_{t-} dV_t + [V, X]_\tau - [P, X]_\tau \middle| \mathcal{H}_0 \right), \end{aligned}$$

over all *admissible* (H, λ, X).

That is those processes and price functions $(H, \lambda) \in \mathfrak{H}$ satisfying:

(A1) $X_t = M_t + A_t + \int_0^t \theta_s \mathrm{d}s$, for all $t \geq 0$,where M is a continuous $\mathbb{H}$-martingale, A a finite variation $\mathbb{H}$-predictable process with $A_t = \sum_{0<s\leq t} (X_s - X_{s-})$, and θ a càdlàg $\mathbb{H}$-adapted process

(A2)

$\mathbb{E}\left(\left(\int_0^\tau (\partial_2 H(s, \xi_s))^2 + (H(s, \xi_s))^2 + V_s^2\right)\left(\sigma_Z^2(s)\mathrm{d}s + \sigma_M^2(s)\mathrm{d}s\right)\right) < \infty$, where $\sigma_M^2(s) := \frac{\mathrm{d}[M,M]_s}{\mathrm{d}s}$,

(A3) $\mathbb{E}\left(\int_0^\tau (\partial_2 H(s, \xi_s) + H(s, \xi_s) + V_s) |\theta_s| \mathrm{d}s\right) < \infty$

(A4) $\mathbb{E}\left(\sum_0^\tau \partial_2 H(s, \xi_{s-}) |\Delta X_s|\right) < \infty$, $\Delta X_s := X_s - X_{s-}$

(A5) $\mathbb{E}\left(\int_0^\tau (H^{-1}(\tau, \cdot)(V_{s-}))^2 + |Z_s|^2 + |X_{s-}|^2)\mathrm{d}[V, V]_s\right) < \infty$

(A6) $\mathbb{E}\left(\int_0^\tau \lambda_s |\partial_{22} H(s, \xi_s)| \left(\sigma_M^2(s) + |\sigma_{M,Z}(s)|\right)\mathrm{d}s\right) < \infty$, where $\sigma_{M,Z}(s) := \frac{\mathrm{d}[M,Z]_s}{\mathrm{d}s}$.

Results

In the following we assume τ such that $\mathbb{E}\left(\int_0^\tau X_t \mathrm{d}V_t \middle| \mathcal{H}_0 \right) = 0$. Hence,

$$\mathbb{E}\left(W_\tau \middle| \mathcal{H}_0 \right) = \mathbb{E}\left(\int_0^\tau (V_{t-} - H(t, \xi_{t-}))\mathrm{d}X_t + [V, X]_\tau - [P, X]_\tau \middle| \mathcal{H}_0 \right).$$

Theorem

Consider an admissible triple (H, λ, X) then if (H, λ, X) is an equilibrium and prices are continuous, on $[0, \tau]$,

$$V_t - H(t, \xi_t) - \frac{\lambda_t}{\eta_t}\mathbb{E}\left(\int_{t\wedge\tau}^\tau \partial_2 H(s, \xi_s)\eta_s \mathrm{d}X_s + [\eta \partial_2 H(\cdot, \xi), X]_{t\wedge\tau}^\tau \middle| \mathcal{H}_t \right) = \quad (1)$$

where

$$\eta_t := \exp\left(\int_0^t \partial_2 H \partial_3 \lambda \mathrm{d}Y_s - \frac{1}{2}\int_0^t (\partial_2 H \partial_3 \lambda)^2 \sigma_Y^2(s)\mathrm{d}s \right).$$

Results

Theorem

Consider an admissible triple (H, λ, X) *then if* (H, λ, X) *is an equilibrium, prices are continuous,* $\tau \equiv T$*, and* $V_t - P_t \neq 0$ *on* $[0, T)$*, then* $\left(\frac{\eta_t}{\lambda_t}\right)_{0 \leq t \leq T}$ *is an* $\mathbb{F}$*-martingale.*

Theorem

Consider an admissible triple (H, λ, X) *then if* (H, λ, X) *is an equilibrium, prices are continuous,* τ *has a density,* $P(\tau > t) > 0$ *and it is independent of the other processes, then if* $V_t - P_t \neq 0$ *on* $[0, \tau)$ $\left(\frac{\eta_t P(\tau > t)}{\lambda_t}\right)_{t \geq 0}$ *is an* $\mathbb{F}$*-martingale.*

A proof for a particular case

Consider the case when $\lambda \geq K > 0$, bounded and C^2 with bounded derivatives and a function of the prices:

$$\lambda_t = \lambda(P_t).$$

Assume also that

$$V_t = V, \mathrm{d}X_t = \theta_t \mathrm{d}t, \tau = T, \mathrm{d}Z_t = \sigma_t \mathrm{d}B_t,$$

where B is an $\mathbb{H}$-Brownian motion, σ bounded, and we assume a price function of the form:

$$P_t = P_0 + \int_0^t \lambda(P_s)\mathrm{d}Y_s,$$

$Y = X + Z$, $X_0 = Y_0 = 0$. Let

$$W_T := \int_0^T (V - P_t)\theta_t \mathrm{d}t$$

We want to maximize $\mathbb{E}\left(W_T \,|\, \mathcal{H}_0 \right)$.

Then if we take $\theta_t^{(\varepsilon)} = \theta_t + \varepsilon\beta_t$,where β is $\mathbb{H}$-adapted,

$$W_T^{(\varepsilon)} - W_T := \int_0^T \left(-P_s^{(\varepsilon)} + P_s\right)\theta_s \mathrm{d}s + \varepsilon \int_0^T (V - P_s)\beta_s \mathrm{d}s.$$

$$\left.\frac{\mathrm{d}\mathbb{E}\left(W_T^{(\varepsilon)})\right)}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} = \mathbb{E}\left(-\int_0^T \left.\frac{\mathrm{d}P_s^{(\varepsilon)}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} \theta_s \mathrm{d}s + \int_0^T (V - P_s)\beta_s \mathrm{d}s\right). \tag{2}$$

And

$$\left.\frac{\mathrm{d}P_t^{(\varepsilon)}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} = \int_0^t \lambda'(P_s) \left.\frac{\mathrm{d}P_s^{(\varepsilon)}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} \mathrm{d}Y_s + \int_0^t \lambda(P_s)\beta_s \mathrm{d}s, \qquad \left.\frac{\mathrm{d}P_t^{(\varepsilon)}}{\mathrm{d}\varepsilon}\right|_{t=0} = 0. \tag{3}$$

The solution of (3) is given by

$$F_t := \eta_t \int_0^t \frac{\lambda(P_s)}{\eta_s} \beta_s \mathrm{d}s. \tag{4}$$

In fact

$$\begin{aligned} \mathrm{d}F_t &= \left(\int_0^t \frac{\lambda(P_s)}{\eta_s} \beta_s \mathrm{d}s \right) \mathrm{d}\eta_t + \lambda(P_t)\beta_t \mathrm{d}t \\ &= \eta_t \left(\int_0^t \frac{\lambda(P_s)}{\eta_s} \beta_s \mathrm{d}s \right) \lambda'(P_t) \mathrm{d}Y_t + \lambda(P_t)\beta_t \mathrm{d}t \\ &= F_t \lambda'(P_s) \mathrm{d}Y_s + \lambda(P_t)\beta_t \mathrm{d}t. \end{aligned}$$

Then plugging (4) into (2) we have

$$\begin{aligned}
&\left.\frac{\mathrm{d}\mathbb{E}\left(W_T^{(\varepsilon)})\right)}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} \\
= \ & \mathbb{E}\left(-\int_0^T \left.\frac{\mathrm{d}P_s^{(\varepsilon)}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} \theta_s \mathrm{d}s + \int_0^T (V - P_s)\beta_s \mathrm{d}s\right) \\
= \ & \mathbb{E}\left(-\int_0^T \eta_s \left(\int_0^s \frac{\lambda(P_u)}{\eta_u}\beta_u \mathrm{d}u\right)\theta_s \mathrm{d}s + \int_0^T (V - P_s)\beta_s \mathrm{d}s\right) \\
= \ & \mathbb{E}\left(-\int_0^T \frac{\lambda(P_s)}{\eta_s}\left(\int_s^T \eta_u\theta_u \mathrm{d}u\right)\beta_s \mathrm{d}s + \int_0^T (V - P_s)\beta_s \mathrm{d}s\right) \\
= \ & \mathbb{E}\left(\int_0^T \left\{V - P_s - \frac{\lambda(P_s)}{\eta_s}\left(\int_s^T \eta_u\theta_u \mathrm{d}u\right)\right\}\beta_s \mathrm{d}s\right).
\end{aligned}$$

Then, by a usual argument $\left.\frac{\mathrm{d}\mathbb{E}\left(W_T^{(\varepsilon)}\right)}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} = 0$ implies that

$$V - P_t - \frac{\lambda(P_t)}{\eta_t}\mathbb{E}\left(\int_t^T \eta_s \theta_s \mathrm{d}s \middle| \mathcal{H}_t\right) = 0$$

And we have (1).
Now, since Z is an $\mathbb{H}$-martingale,

$$\begin{aligned} 0 &= \frac{(V - P_t)\,\eta_t}{\lambda_t} - \mathbb{E}\left(\int_t^T \eta_s \mathrm{d}Y_s \middle| \mathcal{H}_t\right) \\ &= \frac{(V - P_t)\,\eta_t}{\lambda_t} - \mathbb{E}\left(\int_t^T \frac{\eta_s}{\lambda_s} \mathrm{d}P_s \middle| \mathcal{H}_t\right). \end{aligned} \quad (5)$$

and we have that

$$\frac{P_T\eta_T}{\lambda_T} - \frac{P_t\eta_t}{\lambda_t} = \int_t^T \frac{\eta_s}{\lambda_s}\mathrm{d}P_s + \int_t^T P_s\mathrm{d}\left(\frac{\eta_s}{\lambda_s}\right) + \left[P, \frac{\eta}{\lambda}\right]_t^T,$$

$$\begin{aligned}
\mathrm{d}\left(\frac{\eta_s}{\lambda_s}\right) &= \eta_s\mathrm{d}\left(\frac{1}{\lambda_s}\right) + \frac{1}{\lambda_s}\mathrm{d}\eta_s + \mathrm{d}\left[\eta, \frac{1}{\lambda}\right]_s \\
&= -\eta_s\frac{\lambda_s'}{\lambda_s^2}\lambda_s\mathrm{d}Y_s - \frac{1}{2}\eta_s\frac{\lambda_s''\lambda_s^2 - 2\left(\lambda_s'\right)^2\lambda_s}{\lambda_s^4}\lambda_s^2\sigma_s^2\mathrm{d}s \\
&\quad + \frac{1}{\lambda_s}\eta_s\lambda_s'\mathrm{d}Y_s \\
&\quad - \frac{\left(\lambda_s'\right)^2}{\lambda_s}\sigma_s^2\eta_s\mathrm{d}s \\
&= -\frac{1}{2}\frac{\lambda_s''\lambda_s - 2\left(\lambda_s'\right)^2}{\lambda_s}\sigma_s^2\eta_s\mathrm{d}s - \frac{\left(\lambda_s'\right)^2}{\lambda_s}\sigma_s^2\mathrm{d}s \\
&= -\frac{1}{2}\lambda_s''\eta_s\sigma_s^2\mathrm{d}s. \qquad (6)
\end{aligned}$$

Then $\left(\frac{\eta_s}{\lambda_s}\right)$ is absolutely continuous and therefore $\left[P, \frac{\eta}{\lambda}\right] = 0$.
Consequently

$$\frac{P_T \eta_T}{\lambda_T} - \frac{P_t \eta_t}{\lambda_t} = \int_t^T \frac{\eta_s}{\lambda_s} \mathrm{d}P_s - \frac{1}{2} \int_t^T P_s \lambda_s'' \eta_s \sigma_s^2 \mathrm{d}s.$$

And plugging it in (5)

$$0 = \frac{(V - P_t)\, \eta_t}{\lambda_t} + \frac{P_t \eta_t}{\lambda_t} - \mathbb{E}\left(\frac{P_T \eta_T}{\lambda_T} \middle| \mathcal{H}_t \right) - \frac{1}{2} \mathbb{E}\left(\int_t^T P_s \lambda_s'' \eta_s \sigma_s^2 \mathrm{d}s \middle| \mathcal{H}_t \right),$$

therefore

$$
\begin{aligned}
0 &= V\mathrm{d}\left(\frac{\eta_t}{\lambda_t}\right) - \mathrm{d}\mathbb{E}\left(\left.\frac{P_T\eta_T}{\lambda_T}\right| \mathcal{H}_t\right) - \frac{1}{2}\mathrm{d}\mathbb{E}\left(\left.\int_0^T P_s\lambda_s''\eta_s\sigma_s^2\mathrm{d}s\right| \mathcal{H}_t\right) \\
&\quad + \frac{1}{2}P_s\lambda_s''\eta_s\sigma_s^2\mathrm{d}s.
\end{aligned}
$$

By (6)

$$
\begin{aligned}
0 &= -\frac{1}{2}\left(V - P_s\right)\lambda_s''\eta_s\sigma_s^2\mathrm{d}s - \mathrm{d}\mathbb{E}\left(\left.\frac{P_T\eta_T}{\lambda_T}\right| \mathcal{H}_t\right) \\
&\quad - \frac{1}{2}\mathrm{d}\mathbb{E}\left(\left.\int_0^T P_s\lambda_s''\eta_s\sigma_s^2\mathrm{d}s\right| \mathcal{H}_t\right),
\end{aligned}
$$

so

$$\mathrm{d}\mathbb{E}\left(\left.\frac{P_T\eta_T}{\lambda_T}\right|\mathcal{H}_t\right)=-\frac{1}{2}\mathrm{d}\mathbb{E}\left(\left.\int_0^T P_s\lambda_s''\eta_s\sigma_s^2\mathrm{d}s\right|\mathcal{H}_t\right)$$

and

$$-\frac{1}{2}\left(V-P_s\right)\lambda_s''\eta_s\sigma_s^2\mathrm{d}s=0,$$

then if $V-P_t\neq 0$ this implies that

$$\lambda_s''=0$$

and that $\left(\frac{\eta_t}{\lambda_t}\right)$ is a trivial martingale.

Notice that, since prices are $\mathbb{F}$- martingales, λ is also an $\mathbb{F}$- martingale. Also from (6)

$$\begin{aligned}
\lambda_t &= a\eta_t \\
&= a\exp\left(\int_0^t \lambda'(P_s)\mathrm{d}Y_s - \frac{1}{2}\int_0^t \left(\lambda'(P_s)\right)^2 \sigma_s^2 \mathrm{d}s\right) \\
&= a\exp\left(cY_t - \frac{c^2}{2}\int_0^t \sigma_s^2 \mathrm{d}s\right) \\
&= cP_t + b,
\end{aligned}$$

a being a positive constant. We also have that, if $c \neq 0$,

$$P_t - P_0 = a\int_0^t \exp\left(cY_s - \frac{c^2}{2}\int_0^s \sigma_u^2 \mathrm{d}u\right)\mathrm{d}Y_s = \frac{a}{c}\exp\left(cY_t - \frac{c^2}{2}\int_0^t \sigma_s^2 \mathrm{d}s\right)$$

Then taking $P_0 = \frac{a}{c}$

$$P_t = P_0 \exp\left(cY_t - \frac{c^2}{2}\int_0^t \sigma_s^2 \mathrm{d}s\right),$$

and $b = 0$ (Black-Scholes model). If $c = 0$ then λ_t is constant and $P_t = P_0 + bY_t$, but then prices could be negative (Bachelier model).

Exemple

(Back-Baruch, 2004) If the horizon is random with $\tau \sim \exp(r)$, then (5) becomes

$$0 = \frac{(V - P_t)\,\eta_t P(\tau > t)}{\lambda_t} - \mathbb{E}\left(\int_t^\infty \frac{\eta_s P(\tau > s)}{\lambda_s} \mathrm{d}P_s \middle| \mathcal{H}_t \right),$$

and since

$$\begin{aligned} & \frac{P_T \eta_T P(\tau > T)}{\lambda_T} - \frac{P_t \eta_t P(\tau > t)}{\lambda_t} \\ = \; & \int_t^T \frac{\eta_s P(\tau > s)}{\lambda_s} \mathrm{d}P_s + \int_t^T P_s \mathrm{d}\left(\frac{\eta_s P(\tau > s)}{\lambda_s} \right) + \left[P, \frac{\eta(\tau > s)}{\lambda} \right]_t^T, \end{aligned}$$

M*utas mutandis we have that*

$$\mathrm{d}\left(\frac{\eta_s P(\tau > s)}{\lambda_s}\right) = -\left(\frac{1}{2}\lambda_s''\sigma_s^2 + \frac{r}{\lambda_s}\right)\eta_s P(\tau > s)\mathrm{d}s$$

and

$$\frac{1}{2}\lambda_s''\sigma_s^2 + \frac{r}{\lambda_s} = 0.$$

that is the Equation (1.21) in Back-Baruch (2004).

Exemple

(Collin-Dufresne and Fos (2014))
If we take strategies X, s.t.

$$\mathrm{d}X_t = \beta_t (V - P_t)\mathrm{d}t$$

we have

$$\mathrm{d}P_t = \lambda_t \beta_t (V - P_t)\mathrm{d}t + \lambda_t \sigma_z(t)\mathrm{d}B_t^z.$$

Where we assume that $\sigma_z(t)$ is a process independent of the other processes, and where V is also Gaussian. Let denote $m_t = \mathbb{E}(V|\mathcal{F}_t^Y)$, if we apply filtering results, we have

$$\mathrm{d}m_t = \frac{\Sigma_t \beta_t}{\lambda_t \sigma_z^2(t)}(\mathrm{d}P_t - \lambda_t \beta_t (m_t - P_t)\mathrm{d}t), \quad \frac{\mathrm{d}}{\mathrm{d}t}\Sigma_t = -\frac{(\Sigma_t \beta_t)^2}{\sigma_z^2(t)},$$

where Σ_t is the filtering error. Now, we can recover the identity $P_t = m_t$ (*rationality of prices* condition), if and only if we impose $\Sigma_t \beta_t = \lambda_t \sigma_z^2(t)$ (we take $P_0 = m_0 = E(V)$) .

Example

So we have

$$\lambda_t = \frac{\Sigma_t \beta_t}{\sigma_z^2(t)}.$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\Sigma_t = -\frac{(\Sigma_t \beta_t)^2}{\sigma_z^2(t)} = -\lambda_t^2 \sigma_z^2(t).$$

We look for a solution of the form,

$$\lambda_t = \sqrt{\frac{\Sigma_t}{G_t}}$$

where (G_t) has to be a process such that $\left(\frac{1}{\lambda_t}\right)$ is a martingale.

Example

Since

$$\begin{aligned} \mathrm{d}\frac{1}{\lambda_t} &= \frac{1}{\sqrt{\Sigma_t}}\mathrm{d}\sqrt{G_t} - \frac{\sqrt{G_t}}{2\Sigma_t^{3/2}}\mathrm{d}\Sigma_t \\ &= \frac{1}{\sqrt{\Sigma_t}}\mathrm{d}\sqrt{G_t} + \frac{1}{2\sqrt{\Sigma_t G_t}}\sigma_z^2(t)\mathrm{d}t \\ &= \frac{1}{\sqrt{\Sigma_t}}\left(\mathrm{d}\sqrt{G_t} + \frac{1}{2\sqrt{G_t}}\sigma_z^2(t)\mathrm{d}t\right). \end{aligned}$$

If we take (Equation 5 in Collin-Dufresne and Fos (2014))

$$\sqrt{G_t} = \mathbb{E}\left(\int_t^T \frac{\sigma_z^2(t)}{2\sqrt{G_t}}\mathrm{d}t \,\middle|\, \sigma_z^2(s), 0 \le s \le t\right),$$

this is consistent with the condition that $\left(\frac{1}{\lambda_t}\right)$ is a martingale since

$$\mathrm{d}\sqrt{G_t} + \frac{1}{2\sqrt{G_t}}\sigma_z^2(t)\mathrm{d}t = \mathrm{d}\mathbb{E}\left(\int_0^T \frac{\sigma_z^2(t)}{2\sqrt{G_t}}\mathrm{d}t \,\middle|\, \sigma_z^2(s), 0 \le s \le t\right) = \mathrm{d}M_t.$$

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