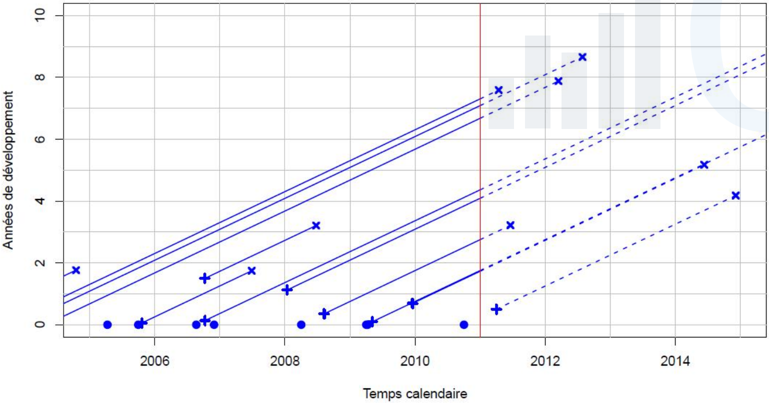
Sarmanov Family of Bivariate Distributions for Multivariate Loss Reserving Analysis

Second International Congress on Actuarial Science and Quantitative Finance

Jean-Philippe Boucher (with Anas Abdallah and Hélène Cossette)

*Quantact / Département de mathématiques
Université du Québec à Montréal, Canada
http://jpboucher.uqam.ca/



Années de développement
0
2
4
6
8
10
2006
2008
2010
2012
2014
Temps calendaire

Quantact

Notations

- $C_{i,j}$ is the cumulative amount up to period j, of claims occurring in accident year i ;
- $C_{i,j}$ represents payment or incurred amounts.

These notations are illustrated in the following loss triangle:

$$C = \begin{pmatrix} C_{1,1} & C_{1,2} & \cdots & C_{1,n-1} & C_{1,n} \\ C_{2,1} & C_{2,2} & \cdots & C_{2,n-1} & \\ \vdots & \vdots & \ddots & & \\ C_{n-1,1} & C_{n-1,2} & & & \\ C_{n,n} & & & & \end{pmatrix}$$

Modeling Run-off Triangles

Classic run-off triangles models:

- Bornhuetter-Ferguson;
- London-Chain;
- etc.

Instead of using these deterministic methods, we often used stochastic methods which allow us to compute the distribution of the reserve, and for example the variance.

Quantact

Dependence between Run-off Triangles

1. Non-Parametric Approaches:
   - Multivariate Mack;
   - Multivariate additive model;
   - etc.

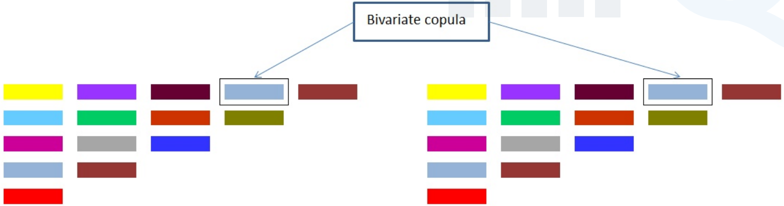
2. Parametric Approaches:
   - Factor models;
   - Copulas;
   - Random Effects.

UQÀM
Université du Québec à Montréal

Recent Approaches for Dependence structures

1. Pair-Wise Dependence (Frees and Shi, 2011);
2. Multivariate Gaussian Copula Dependence;
3. Hierarchical Archimedean Copulas Dependence (Abdallah, Boucher and Cossette, 2015).

Pair-Wise Dependence



Pair-Wise Dependence by Frees and Shi, 2011

Frees and Shi selected marginal distributions, and then used several copulas:

1. Independence;
2. Gaussian;
3. Frank.

To obtain MLE estimates, the loglikelihood of the model can be expressed as:

$$L = \sum_{i=1}^{I} \sum_{j=1}^{I-i+1} \log(f_{ij}^{(1)}) + \log(f_{ij}^{(2)}) + \sum_{i=1}^{I} \sum_{j=1}^{I-i+1} \log \; c(F_{ij}^{(1)}, F_{ij}^{(2)}; \theta) \; ,$$

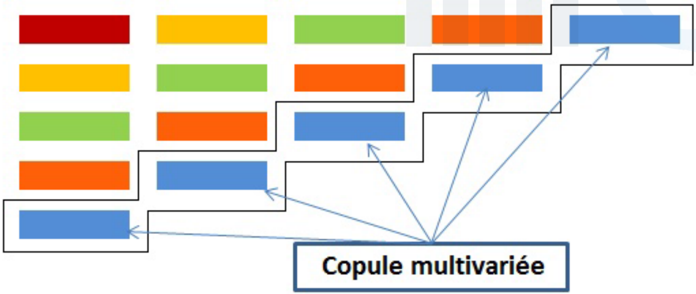
where $c(.)$ denotes the probability density function corresponding to the copula distribution function $C(.)$, $f_{ij}^{(\ell)}$ denotes the density of marginal distribution $F_{ij}^{(\ell)}$, for $\ell = 1, 2$.

Generalization

Instead of supposing a cell-by-cell dependence, we suppose a two-step modeling:

1. A dependence between each cell of a diagonal (i.e. calendar year);
2. A dependence between each diagonal.

Calendar Year Dependence



Calendar Year Dependence

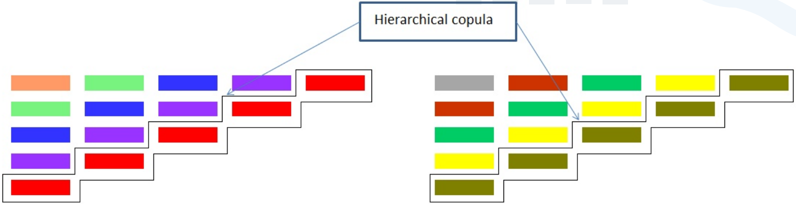
For each line of business, we selected marginal distributions, and used a copula for the dependence.

To obtain MLE estimates, the loglikelihood of the model can be expressed as:

$$L = \sum_{i=1}^{I} \sum_{j=1}^{I-i+1} \log(f_{ij}) + \sum_{t=2}^{n} \log \; c\left(F_{t-j+1,j}, \ldots, F_{1,t}; \theta\right)_{j=1,\ldots,t}$$

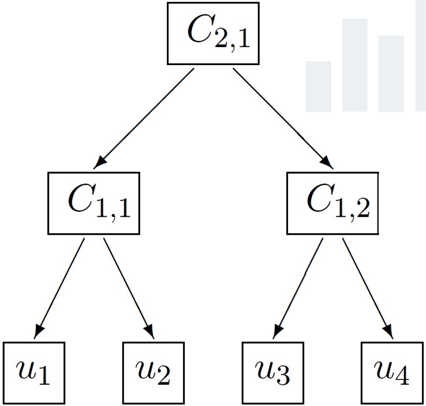
where f denotes the density of marginal distribution F, and $c(.)$ the probability density function corresponding to the copula distribution function $C(.)$.

Hierarchical Archimedean Copulas Dependence



Quantact

Hierarchical Archimedean Copulas Dependence (2)



UQÀM
Université du Québec à Montréal

Conditions for Hierarchical Dependence with Copulas

Mathematically, the conditions that have to be verified by a hierarchical Archimedean copula are summarized as follows:

1. All inverse generator functions $\phi_{h,k}^{-1}$ are completely monotone.
2. The composite $\phi_{h+1,k'} \circ \phi_{h,k}^{-1}$ are convex functions for all $h = 0, ..., H$ and $k = 1, ..., n_h$, $k' = 1, ..., n_{h+1}$ such that $C_{h,k} \in C_{h+1,k'}$.

The log-likelihood function of the hierarchical model can be written as:

$$L = \sum_{\ell=1}^{2}\sum_{i=1}^{I}\sum_{j=1}^{I-i+1}\log(f_{ij}^{(\ell)}) + \sum_{t=2}^{n}\log\left(c_{2,1}\left(F_{t,1}^{(1)}, F_{t-1,1}^{(1)}, \ldots, F_{1,t}^{(1)}, F_{t,1}^{(2)}, F_{t-1,1}^{(2)}, \ldots, F_{1,t}^{(2)}\right)\right),$$

where $c_{2,1}$ denotes the density of a hierarchical Archimedean copula.

Density of a Hierarchical Copulas

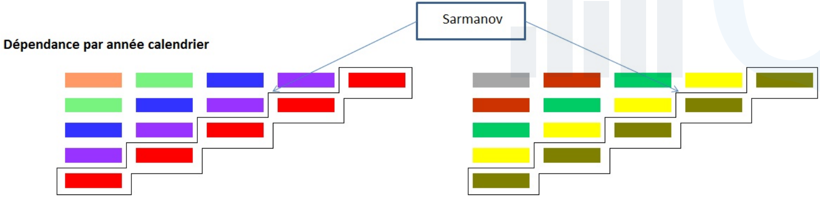
The density of a hierarchical Archimedean copula, which is obtained by differentiating the copula, i.e:

$$c_{2,1}(\mathbf{u}) = \frac{\partial^{2t} C_{2,1}(\mathbf{u})}{\partial u^{(1)}_{t-j+1,j} \cdots \partial u^{(1)}_{1,t} \partial u^{(2)}_{t-j+1,j} \cdots \partial u^{(2)}_{1,t}}.$$

As we have 10 diagonals, we need to derive up to 20 times...

A numerically efficient way to evaluate the log-density is presented in Hofert and Pham (2013).

Instead of Copulas Dependence (2)



Approach

As for the hierarchical Archimedean copulas approach, we suppose a two-step modeling:

1. A dependence between each cell of a diagonal: with random effects;
2. A dependence between each diagonal with a joint distribution (Sarmanov distribution).

Random Effects and Calendar Year

Given $\Theta^{(\ell)}$, the losses of the cells of a given calendar are random variables $Y_1^{(\ell)}, \ldots, Y_T^{(\ell)}$ are (conditionally) independent.

The joint distribution can be expressed as:

$$
\begin{aligned}
f(Y_1^{(\ell)}, \ldots, Y_T^{(\ell)}) &= \int f(Y_1^{(\ell)}, \ldots, Y_T^{(\ell)} | \Theta^{(\ell)}) g(\Theta^{(\ell)}) d\Theta^{(\ell)} \\
&= \int \prod_{t=1}^{T} f(Y_t^{(\ell)} | \Theta^{(\ell)}) g(\Theta^{(\ell)}) d\Theta^{(\ell)}
\end{aligned}
$$

For example, we can suppose conjugate distributions:

1. $Y_t^{(1)} \mid \Theta^{(1)} = \theta^{(1)} \sim$ Logn. $\left(\mu_t^{(1)}\theta^{(1)}, \sigma^2\right)$, with $\Theta^{(1)} \sim$ Normal $\left(a, b^2\right)$;
2. $Y_y^{(2)} \mid \Theta^{(2)} = \theta^{(2)} \sim$ Gamma $\left(\phi, \frac{\mu_t^{(2)}}{\theta^{(2)}}\right)$, with $\Theta^{(2)} \sim$ Gamma (α, τ);
3. etc.

Dependence between Triangles

Given $\Theta^{(1)}, \Theta^{(2)}$,, random variables $Y_1^{(1)}, ..., Y_T^{(1)}, Y_1^{(2)}, ..., Y_T^{(2)}$ are (conditionally) independent.

The joint distribution can be expressed as:

$$f(Y_1^{(1)}, ..., Y_T^{(1)}, Y_1^{(2)}, ..., Y_T^{(2)})$$
$$= \int \int f(Y_1^{(1)}, ..., Y_T^{(1)}, Y_1^{(2)}, ..., Y_T^{(2)} | \Theta^{(1)}, \Theta^{(2)}) g(\Theta^{(1)}, \Theta^{(2)}) d\Theta^{(1)} d\Theta^{(2)}$$
$$= \int \int \left[\prod_{t=1}^{T} f(Y_t^{(1)} | \Theta^{(1)}) \right] \left[\prod_{t=1}^{T} f(Y_t^{(2)} | \Theta^{(2)}) \right] g(\Theta^{(1)}, \Theta^{(2)}) d\Theta^{(1)} d\Theta^{(2)}$$

Quantact

Independence between Triangles

When we suppose independence between random effects, the joint distribution can be expressed as:

$$
\begin{aligned}
& f(Y_1^{(1)}, ..., Y_T^{(1)}, Y_1^{(2)}, ..., Y_T^{(2)}) \\
&= \int\int \left[\prod_{t=1}^{T} f(Y_t^{(1)}|\Theta^{(1)})\right]\left[\prod_{t=1}^{T} f(Y_t^{(2)}|\Theta^{(2)})\right] g(\Theta^{(1)})g(\Theta^{(2)})d\Theta^{(1)}d\Theta^{(2)} \\
&= \int \left[\prod_{t=1}^{T} f(Y_t^{(1)}|\Theta^{(1)})\right] g(\Theta^{(1)})d\Theta^{(1)} \times \int \left[\prod_{t=1}^{T} f(Y_t^{(2)}|\Theta^{(2)})\right] g(\Theta^{(2)})d\Theta^{(2)} \\
&= \qquad f(Y_1^{(1)}, ..., Y_T^{(1)}) \qquad \times \qquad f(Y_1^{(2)}, ..., Y_T^{(2)})
\end{aligned}
$$

Quantact

Sarmanov Distribution

Sarmanov's bivariate distribution was introduced by Sarmanov (1966).

The joint distribution can then be expressed as:

$$g^{S}\left(\Theta^{(1)},\Theta^{(2)}\right)=g^{(1)}\left(\Theta^{(1)}\right)g^{(2)}\left(\Theta^{(2)}\right)\left(1+\omega\psi^{(1)}\left(\Theta^{(1)}\right)\psi^{(2)}\left(\Theta^{(2)}\right)\right),$$

with, for $\ell = 1, 2$:

- $\psi^{(\ell)}\left(\theta^{(\ell)}\right) = \exp\left(-\Theta^{(\ell)}\right) - L_{(\ell)}(1)$,
- $g(\Theta^{(\ell)})$ is the marginal density of the random effects of triangle ℓ.

UQÀM
Université du Québec à Montréal

Sarmanov Density

If we suppose $\Theta^{(1)} \sim$ Normal $\left(a, b^2\right)$ and $\Theta^{(2)} \sim$ Gamma (α, τ), we obtain:

$$
\begin{aligned}
& g^S\left(\theta^{(1)}, \theta^{(2)}\right) \\
= \; & g^{(1)}\left(\theta^{(1)}; a, b^2\right) g^{(2)}\left(\theta^{(2)}; \alpha, \tau\right)\left(1+\omega \exp\left(-a+\frac{b^2}{2}\right)(1+\tau)^{-\alpha}\right) \\
& + g^{(1)}\left(\theta^{(1)}; a-b^2, b^2\right) g^{(2)}\left(\theta^{(2)}; \alpha, \frac{\tau}{1+\tau}\right) \omega \exp\left(-a+\frac{b^2}{2}\right)(1+\tau)^{-\alpha} \\
& - g^{(1)}\left(\theta^{(1)}; a-b^2, b^2\right) g^{(2)}\left(\theta^{(2)}; \alpha, \tau\right) \omega \exp\left(-a+\frac{b^2}{2}\right)(1+\tau)^{-\alpha} \\
& - g^{(1)}\left(\theta^{(1)}; a, b^2\right) g^{(2)}\left(\theta^{(2)}; \alpha, \frac{\tau}{1+\tau}\right) \omega \exp\left(-a+\frac{b^2}{2}\right)(1+\tau)^{-\alpha}
\end{aligned}
$$

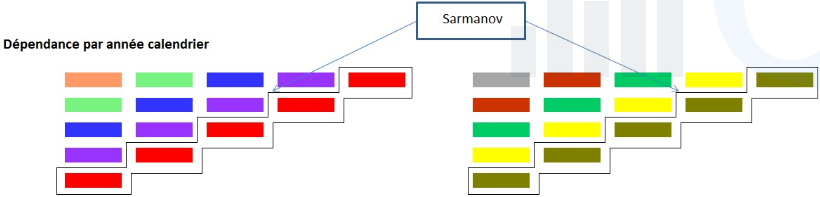
Joint density function

If the losses are conditional lognormal for line 1, and conditional gamma for line 2, we can express the joint distribution as:

$$
\begin{aligned}
& f_{\mathbf{Y}_t^{(1)},\mathbf{Y}_t^{(2)}}\left(\mathbf{y}_t^{(1)},\mathbf{y}_t^{(2)}\right) \\
= \; & f_{\mathbf{Y}_t^{(1)}}\left(\mathbf{y}_t^{(1)}; a, b^2\right) f_{\mathbf{Y}_t^{(2)}}\left(\mathbf{y}_t^{(2)}; \alpha, \tau\right)\left(1 + \omega \exp\left(-a + \frac{b^2}{2}\right)(1+\tau)^{-\alpha}\right) \\
& + f_{\mathbf{Y}_t^{(1)}}\left(\mathbf{y}_t^{(1)}; a - b^2, b^2\right) f_{\mathbf{Y}_t^{(2)}}\left(\mathbf{y}_t^{(2)}; \alpha, \frac{\tau}{1+\tau}\right)\omega \exp\left(-a + \frac{b^2}{2}\right)(1+\tau)^{-\alpha} \\
& - f_{\mathbf{Y}_t^{(1)}}\left(\mathbf{y}_t^{(1)}; a - b^2, b^2\right) f_{\mathbf{Y}_t^{(2)}}\left(\mathbf{y}_t^{(2)}; \alpha, \tau\right)\omega \exp\left(-a + \frac{b^2}{2}\right)(1+\tau)^{-\alpha} \\
& - f_{\mathbf{Y}_t^{(1)}}\left(\mathbf{y}_t^{(1)}; a, b^2\right) f_{\mathbf{Y}_t^{(2)}}\left(\mathbf{y}_t^{(2)}; \alpha, \frac{\tau}{1+\tau}\right)\omega \exp\left(-a + \frac{b^2}{2}\right)(1+\tau)^{-\alpha}.
\end{aligned}
$$

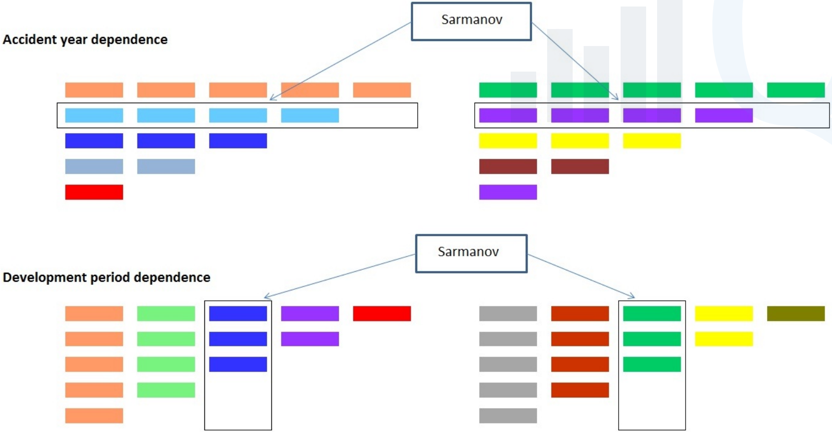
Quantact

Calendar Year Dependence



UQÀM
Université du Québec à Montréal

Illustration



Total unpaid losses incorporate past information!
The posterior bivariate joint density function of the couple $\left(\Theta_t^{(1)}, \Theta_t^{(2)}\right)$ conditioned on $\left(\mathbf{Y}_\mathbf{t}^{(1)}, \mathbf{Y}_\mathbf{t}^{(2)}\right)$ is given by:

$$u^S(\theta_t^{(1)}, \theta_t^{(2)} \mid \mathbf{y}_\mathbf{t}^{(1)}, \mathbf{y}_\mathbf{t}^{(2)}) = \frac{f(\mathbf{y}_\mathbf{t}^{(1)}, \mathbf{y}_\mathbf{t}^{(2)} \mid \theta_t^{(1)}, \theta_t^{(2)}) u^S(\theta_t^{(1)}, \theta_t^{(2)})}{f_{\mathbf{Y}_\mathbf{t}^{(1)}, \mathbf{Y}_\mathbf{t}^{(2)}}(y_1^{(1)}, \ldots, y_t^{(1)}, y_1^{(2)}, \ldots, y_t^{(2)})}.$$

Consequently, we get:

$$\begin{aligned} u^S(\theta_t^{(1)}, \theta_t^{(2)} \mid \Im_t) = \; & C_1 \times n\left(\theta_t^{(1)}; a_{post}, b_{post}^2\right) g\left(\theta_t^{(2)}; \alpha_{post}, \tau_{post}\right) \\ & + C_2 \times n\left(\theta_t^{(1)}; a_{post}^{'}, b_{post}^2\right) g\left(\theta_t^{(2)}; \alpha_{post}, \tau_{post}^{'}\right) \\ & - C_3 \times n\left(\theta_t^{(1)}; a_{post}^{'}, b_{post}^2\right) g\left(\theta_t^{(2)}; \alpha_{post}, \tau_{post}\right) \\ & - C_4 \times n\left(\theta_t^{(1)}; a_{post}, b_{post}^2\right) g\left(\theta_t^{(2)}; \alpha_{post}, \tau_{post}^{'}\right) \end{aligned}$$

With constants...

$$a_{post} = \frac{\sum_{k=1}^{t} \log(y_k^{(1)})\mu_k^{(1)} b^2 + a\sigma^2}{\sum_{k=1}^{t} \mu_k^{(1)2} b^2 + \sigma^2}, \; a'_{post} = \frac{\sum_{k=1}^{t} \log(y_k^{(1)})\mu_k^{(1)} b^2 + \left(a - b^2\right)\sigma^2}{\sum_{k=1}^{t} \mu_k^{(1)2} b^2 + \sigma^2},$$

$$b^2_{post} = \left(\frac{\sum_{k=1}^{t} \mu_k^{(1)2}}{\sigma^2} + \frac{1}{b^2}\right)^{-1}, \; \alpha_{post} = t\phi + \alpha, \; \tau_{post} = \left(\sum_{k=1}^{t} \frac{y_k^{(2)}}{\mu_k^{(2)}} + \frac{1}{\tau}\right)^{-1},$$

$$\tau'_{post} = \left(\sum_{k=1}^{t} \frac{y_k^{(2)}}{\mu_k^{(2)}} + \frac{1}{\tau} + 1\right)^{-1}$$ and

$$C_1 = c^{-1} g^{(1)}\left(\mathbf{y_t}^{(1)}; \theta_t^{(1)} \mu_{i,j}^{(1)}, \sigma^2, a, b^2\right) g^{(2)}\left(\mathbf{y_t}^{(2)}; \frac{\mu_{i,j}^{(2)}}{\theta_t^{(2)}}, \phi, \alpha, \tau\right)\left(1 + \omega \exp\left(-a + \frac{b^2}{2}\right)(1+\tau)^{-\alpha}\right);$$

$$C_2 = c^{-1} \omega g^{(1)}\left(\mathbf{y_t}^{(1)}; \theta_t^{(1)} \mu_{i,j}^{(1)}, \sigma^2, a - b^2, b^2\right) g^{(2)}\left(\mathbf{y_t}^{(2)}; \frac{\mu_{i,j}^{(2)}}{\theta_t^{(2)}}, \phi, \alpha, \frac{\tau}{1+\tau}\right) \exp\left(-a + \frac{b^2}{2}\right)(1+\tau)^{-\alpha};$$

$$C_3 = c^{-1} \omega g^{(1)}\left(\mathbf{y_t}^{(1)}; \theta_t^{(1)} \mu_{i,j}^{(1)}, \sigma^2, a - b^2, b^2\right) g^{(2)}\left(\mathbf{y_t}^{(2)}; \frac{\mu_{i,j}^{(2)}}{\theta_t^{(2)}}, \phi, \alpha, \tau\right) \exp\left(-a + \frac{b^2}{2}\right)(1+\tau)^{-\alpha};$$

$$C_4 = c^{-1} \omega g^{(1)}\left(\mathbf{y_t}^{(1)}; \theta_t^{(1)} \mu_{i,j}^{(1)}, \sigma^2, a, b^2\right) g^{(2)}\left(\mathbf{y_t}^{(2)}; \frac{\mu_{i,j}^{(2)}}{\theta_t^{(2)}}, \phi, \alpha, \frac{\tau}{1+\tau}\right) \exp\left(-a + \frac{b^2}{2}\right)(1+\tau)^{-\alpha}.$$

Estimation results

Fit Statistics	Dependence			
	PWD	Dev. period	Calendar year	Accident year
Log-Likelihood	350.5	376.4	396.4	402.3
AIC	-618.9	-669.0	-708.9	-720.8
BIC	-508.3	-656.2	-696.2	-708.1

Table: Fit Statistics of PWD model vs Independent lob's with random effects

Quantact

Estimated Reserves

Reserves estimation	Dependence			
	PWD	Dev. period	Calendar year	Accident year
Personal	6,423,180	6,547,988	6,476,093	6,586,506
Commercial	495,989	504,928	551,478	445,151
Total	6,919,169	7,052,916	7,027,571	7,031,658

Table: Reserves estimation with different models

UQÀM
Université du Québec à Montréal

Mean Square Error Prediction= MSEP

The MSEP is a combination of process error and estimation error:

$$
\begin{aligned}
MSEP[\widehat{R_{tot}}] &\approx E[(R_{tot} - E[R_{tot}])^2] + E[(\widehat{R_{tot}} - E[\widehat{R_{tot}}])^2] \\
&= \underbrace{Var[R_{tot}]}_{\text{Process error}^2} + \underbrace{Var[\widehat{R_{tot}}]}_{\text{Estimation error}^2} .
\end{aligned}
$$

Model	**Reserve**	$\sqrt{MSEP}$
Sarmanov	7,027,571	347,947
Mack	6,925,951	334,929

Table: Sarmanov vs Mack

Bootstrap procedure

1. From the parameters estimated on the data, we generate new realizations of an upper triangle ;

2. Starting from these new triangles, we obtain new parameters estimates ;

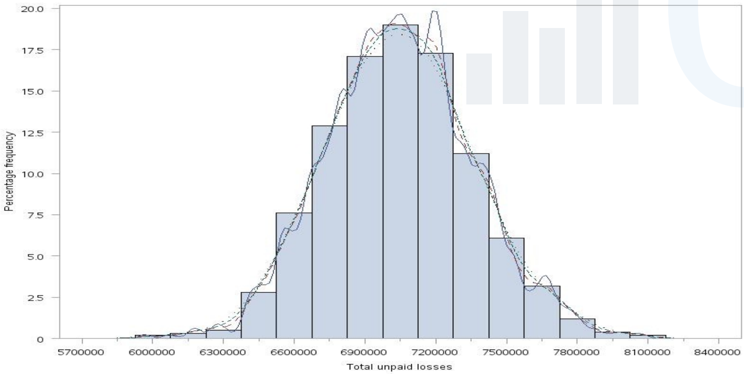
3. Starting from these new estimates, we calculate the projected mean reserve (expected reserve);

or

3*. Starting from these new estimates, we simulate the lower triangle (simulated reserve);

4. We repeat the above steps N times.

Predictive Distribution



Quantact

Risk Capital Analysis

Risk measure	TVaR (80%)	TVaR (85%)	TVaR (90%)	TVaR (95%)	TVaR (99%)
Silo - random effects	7,671,066	7,755,618	7,862,446	8,041,361	8,441,168
Silo - independent	7,582,963	7,656,635	7,760,671	7,922,635	8,259,798
Sarmanov	7,491,092	7,542,301	7,609,383	7,720,910	7,910,013
Risk capital					
Silo - random effects	623,135	707,686	814,515	993,429	1,393,237
Silo - independent	535,032	608,703	712,739	874,704	1,211,866
Sarmanov	443,160	494,369	561,451	672,979	862,082
Gain					
vs independent	17.17%	18.78%	21.23%	23.06%	28.86%
vs random effect	28.88%	30.14%	31.07%	32.26%	38.12%

Table: Risk capital estimation with different scenarios

UQÀM
Université du Québec à Montréal

Conclusion

Advantages of the Sarmanov Approach:

1. Statistical inference is simple;
2. Calendar year, accident year and development year modeling can be used (with *a posteriori* distribution)
3. Moments of the model can be easily calculated.

Extensions:

- Dynamic Random Effects
- Multivariate generalization
- Incorporating expert opinion into a stochastic model