Optimal market dealing under constraints

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Motivations : Liquidity costs for price takers

- Price takers trade only through market orders and pay liquidity costs
  - Transaction costs due to bid-ask spread:
    - → Shreve and Soner (1994); Korn (1998); Framstad, Oksendal and Sulem (2001), ...
  - Price impact for large trades: Almgren and Chriss (2001)
    - → Supply curves: Cetin, Jarrow, Protter (2004); Alfonsi, Fruth and Schied (2010), ...
    - → Impact functions: Bank and Baum (2004); Ly Vath, Mnif and Pham (2007); Kharroubi, Pham (2010); Roch (2011)...
Motivations: Liquidity in limit order book market

- Trade in a limit order book through market and limit order
  → pay less liquidity costs but have some inventory risk.
  → order-driven market (NYSE,...)

- Liquidation problems:
  → Guéant, Lehalle and Tapia (2012); Bayraktar and Ludkovski (2012);
    Bouchard, Lehalle and Dang (2011);...

- Market making/Portfolio management problems:
  → Guilbaud and Pham (2013); C., Ly Vath, Roch and Scotti (2015);...
Motivations: Market dealing under constraints

- trade in a dealer market
  - face liquidity and inventory constraints.
  - quote driven market (Nasdaq, LSE,...)
  - competition with others dealers or act as a single/representative dealer

- A market dealer faces some constraints
  - Provide liquidity
  - Set "reasonable" prices and spread
  - Cash and stock holdings constraints

- Ho, Stoll (1981); Mildenstein and Schleef (1983); Avellaneda and Stoikov (2008);...
1 Model and problem formulation
   - Non controlled model
   - Controlled model
   - An optimal control problem with regime switching

2 Analytical properties and dynamic programming principle
   - Properties of the value functions
   - Dynamic programming principle

3 Viscosity characterization of the objective function

4 Numerical illustrations
We consider a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) with a filtration \(\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}\) satisfying the usual conditions.

Let two Cox processes \(N^a\) and \(N^b\) which intensity processes are defined by two processes \((\lambda^a_t)_{0 \leq t \leq T}\) and \((\lambda^b_t)_{0 \leq t \leq T}\).

We define \(\theta^a_i\) (resp. \(\theta^b_i\)) as the \(i^{th}\) jump time of \(N^a\) (resp. \(N^b\)), which corresponds to the \(i^{th}\) buy (resp. sell) market order.

**Mid price process**: For all \(i \in \mathbb{N}^*\), the dynamics of the process \(P\) is given by
\[
\begin{cases}
    dP_t = 0, & \text{for } \xi_i < t < \xi_{i+1} \\
    P_{\theta^b_i} = P_{\theta^b_i} - \delta \\
    P_{\theta^a_i} = P_{\theta^a_i} + \delta
\end{cases}
\]
Representation of non controlled prices dynamics

Price

\[ \theta^a_1 \quad \theta^b_1 \quad \theta^a_2 \quad \theta^b_2 \]

Ask Price
Bid Price

Time
Limit price process

- **Intensities**: We assume that $\lambda_t^a = \lambda^a(t, P_{t-})$ and $\lambda_t^b = \lambda^b(t, P_{t-})$ where $\lambda^a$ and $\lambda^b$ are deterministic functions.

- **Mid price infinitesimal generator**: For $f \in C^2(\mathbb{R})$, we have

$$
Lf(t, p) = \lambda^a(t, p) (f(p + \delta) - f(p)) + \lambda^b(t, p) (f(p - \delta) - f(p)).
$$

$$
= \delta \left( \lambda^a - \lambda^b \right) (t, p)f'(p) + \frac{\delta^2}{2} \left( \lambda^a + \lambda^b \right) (t, p)f''(p) + o(\delta^2).
$$

- **Limit process**: Under the following assumption:

$$
\lim_{\delta \to 0} \delta \left( \lambda^a - \lambda^b \right) (t, p) = \mu(t, p) \text{ and } \lim_{\delta \to 0} \left( \lambda^a + \lambda^b \right) (t, p) = \sigma(t, p)^2,
$$

the laws of the process $P$ converges weakly to the law of the diffusion process with coefficients $\mu$ and $\sigma$. 
Market making strategies

- We consider a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) with a filtration \(\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}\) satisfying the usual conditions.

- When the \(i^{\text{th}}\) buying (resp. selling) order arrives at the \(\mathbb{F}\)-stopping time \(\theta^a_i\) (resp. \(\theta^b_i\)):
  
  - **Provide liquidity**: The market maker has to sell (resp. buy) an asset at the ask (resp. bid) price denoted by \(P^a\) (resp. \(P^b\)).
  
  - **Set Bid and Ask prices**: The market maker may either keep the bid and ask prices constant or increase (resp. decrease) one or both of them by some ticks (\(\delta\)).

- We consider a control \(\alpha := (\epsilon_t^a, \epsilon_t^b, \eta_t^a, \eta_t^b)_{0 \leq t \leq T}\) \(\mathbb{F}\)-predictable process where the random variables \(\epsilon_t^a, \epsilon_t^b, \eta_t^a, \eta_t^b\) are valued in \(\{-\chi_{\text{min}}, \ldots, \chi_{\text{max}}\}\).
Representation of a market making strategies
**Bid and Ask processes**: For $c \in \{a, b\}$, the dynamics of $P^c$ evolves according to the following equations

\[
\begin{align*}
    dP^c_t &= 0 & \text{for } \xi_i < t < \xi_{i+1} \\
    P^c_{\theta_i^b} &= P^c_{\theta_i^b} - \delta \epsilon^c_{\theta_i^b} \\
    P^c_{\theta_i^a} &= P^c_{\theta_i^a} + \delta \eta^c_{\theta_i^a},
\end{align*}
\]

for $i \in \mathbb{N}^*$, where $(\xi_i)_{i \geq 0}$ is the sequence of transaction times.

**Mid price and spread processes**: We set $P := \frac{P^a + P^b}{2}$ and $S := P^a - P^b$. For all $i \in \mathbb{N}^*$, the dynamics of the process $(P, S)$ is given by

\[
\begin{align*}
    dP_t &= 0, & \text{for } \xi_i < t < \xi_{i+1} \\
    P_{\theta_i^b} &= P_{\theta_i^b} - \frac{\delta}{2} (\epsilon^a_{\theta_i^b} + \epsilon^b_{\theta_i^b}) \\
    P_{\theta_i^a} &= P_{\theta_i^a} + \frac{\delta}{2} (\eta^a_{\theta_i^a} + \eta^b_{\theta_i^a}).
\end{align*}
\]

\[
\begin{align*}
    dS_t &= 0, & \text{for } \xi_i < t < \xi_{i+1} \\
    S_{\theta_i^b} &= S_{\theta_i^b} - \delta (\epsilon^a_{\theta_i^b} - \epsilon^b_{\theta_i^b}) \\
    S_{\theta_i^a} &= S_{\theta_i^a} + \delta (\eta^a_{\theta_i^a} - \eta^b_{\theta_i^a}).
\end{align*}
\]
Cash and stock holdings dynamics

**Cash holdings**: We denote by $r > 0$ the instantaneous interest rate. The bank account evolves according to the following equations

$$
\begin{align*}
\frac{dX_t}{dt} &= r X_t \quad \text{for } \xi_i < t < \xi_{i+1}, \\
X_{\theta_i}^b &= X_{\theta_i^b} - P_{\theta_i^b} \\
X_{\theta_i}^a &= X_{\theta_i^a} + P_{\theta_i^a},
\end{align*}
$$

for $i \in \mathbb{N}^*$.

**Stock holdings**: The number of shares held by the market maker at time $t \in [0, T]$ is denoted by $Y_t$, and evolves according to the following equations

$$
\begin{align*}
\frac{dY_t}{dt} &= 0, \quad \text{for } \xi_i < t < \xi_{i+1}, \\
Y_{\theta_i}^b &= Y_{\theta_i^b} + 1 \\
Y_{\theta_i}^a &= Y_{\theta_i^a} - 1
\end{align*}
$$

for $i \in \mathbb{N}^*$.
Regime switching

- **Liquidity regimes**: Let $I$ be a continuous time, time homogeneous, irreducible Markov chain with $m$ states. The generator of the chain $I$ under $P$ is denoted by $A = (\vartheta_{i,j})_{i,j=1,...,m}$. Here $\vartheta_{i,j}$ is the constant intensity of transition of the chain $L$ from state $i$ to state $j$.

- **Market orders arrivals**: Let two Cox processes $N^a$ and $N^b$. The intensity processes associated with $N^a$ and $N^b$ are defined, for $t \geq 0$, by $\lambda^a(t, l_t, P_t, S_t)$ and $\lambda^b(t, l_t, P_t, S_t)$ where $\lambda^a$ and $\lambda^b$ are positive deterministic functions, bounded and defined on $[0, T] \times \{1, ..., m\} \times \frac{\delta}{2} \mathbb{N} \times \delta \mathbb{N}$.

  We define $\theta^a_i$ (resp. $\theta^b_i$) as the $i^{th}$ jump time of $N^a$ (resp. $N^b$), which corresponds to the $i^{th}$ buy (resp. sell) market order.
**Liquidity constraints**: Let $K > 0$, the market maker has to use controls such that

$$P_t - S_t/2 > 0 \quad \text{and} \quad 0 < S_t \leq K \times \delta,$$

for $0 \leq t \leq T$.

**Inventory and cash constraints**: Let $x_{\min} < 0$ and $y_{\min} \leq y_{\max}$. We introduce the following notations:

$$S = (x_{\min}, +\infty) \times \{y_{\min}, \ldots, y_{\max}\} \times \delta\mathbb{N} \times \delta\{1, \ldots, K\},$$

$$S = \{(t, x, y, p, s) \in [0, T] \times S : p - \frac{s}{2} \geq \delta\}.$$

For a control $\alpha$, we define the liquidation time:

$$\tau_{t,i,z,\alpha} := \inf\{u \geq t : X_{u}^{t,i,x,\alpha} \leq x_{\min} \text{ or } Y_{u}^{t,i,y,\alpha} \in \{y_{\min} - 1, y_{\max} + 1\}\}$$

**Admissible strategies**: Let $(t, z) := (t, x, y, p, s) \in S$, the strategy $\alpha = (\epsilon^{a}_{u}, \epsilon^{b}_{u}, \eta^{a}_{u}, \eta^{b}_{u})_{t \leq u \leq T}$ is admissible, if the processes $\epsilon^{a}, \epsilon^{b}, \eta^{a}, \eta^{b}$ are valued in $\{-\chi_{\min}, \ldots, \chi_{\max}\}$ and for all $u \in [t, T]$, $(u, Z_{u}^{t,i,z,\alpha}) \in S$.

We denote by $\mathcal{A}(t, z)$ the set of all these admissible policies.
Objective function

- **Portfolio liquidation**: If the market maker decides (or has) to liquidate her portfolio, then she actually gets

\[ yQ(t, y, p, s) = y(p - \text{sign}(y) \frac{S}{2})f(t, y), \]

where \( f : [0, T] \times \mathbb{R} \to \mathbb{R}_+ \), non-linear in \( y \) and such that

\[ f(t, y) \leq f(t, y') \text{ if } y' \leq y \quad \text{and} \quad yf(t', y) \leq yf(t, y) \text{ if } t' \leq t. \]

- **Utility and penalty functions**: Let \( \gamma > 0 \) and \( U(x) = 1 - e^{-\gamma x} \) on \( \mathbb{R} \). We set

\[ U_L = U_0L \quad \text{where } L(t, x, y, p, s) = x + yQ(t, y, p, s). \]

Let \( g \) a bounded positive function defined on \( \{y_{\min}, ..., y_{\max}\} \).

- **Objective function**: We consider the functions \( (\nu_i)_{i \in \{1, ..., m\}} \) defined on \( S \) by

\[ \nu_i(t, z) := \sup_{\alpha \in A(t, z)} J_i^\alpha(t, z) \]

where we have set

\[ J_i^\alpha(t, z) := \mathbb{E} \left[ U_L(T \wedge \tau^{t, i, z, \alpha}, Z^{t, i, z, \alpha}_{(T \wedge \tau^{t, i, z, \alpha})^-}) - \int_t^{T \wedge \tau^{t, i, z, \alpha}} g(Y_s^{t, i, y, \alpha}) ds \right]. \]
Analytical properties and dynamic programming principle

- Model and problem formulation
- Analytical properties and dynamic programming principle
- Viscosity characterization of the objective function
- Numerical illustrations
Let \((t, z) := (t, x, y, p, s) \in S\). From monotonicity of \(f\),

\[
L(t, z) \geq x_{\text{min}} + y_{\text{min}} f(0, y_{\text{min}})(p - \frac{K\delta}{2}).
\]

**Proposition**

There exist \(C_1, C_2\) and \(C_3\) positive constants such that

\[
1 - C_1 - C_2 e^{C_3p} \leq v_i(t, z) \leq 1, \quad \forall (i, t, z) := (i, t, x, y, p, s) \in \{1, ..., m\} \times S,
\]
Hölder continuity of the criteria functions

Let $i \in \{1, \ldots, m\}$, $(t, z) := (t, x, y, p, s) \in \tilde{S}$ and $(t', x')$ in $[0, T] \times (x_{\min}, +\infty)$. For all $\alpha \in \mathcal{A}(t \wedge t', z)$ such that $\alpha ||[t \wedge t', t \vee t'] = 0$, we have $\alpha \in \mathcal{A}(t, z) \cap \mathcal{A}(t', z')$ with $z' = (x', y, p, s)$ and, if $(t', x')$ is close enough to $(t, x)$, then

$$| J_i^\alpha(t, z) - J_i^\alpha(t', z') | \leq K_2(p) \left( \psi(r e^{rt} | x'(t - t') |) + \psi(x' - x) + | t' - t | \right).$$

where $K_2(p) > 0$ and $\psi$ an Hölder continuous function on $\mathbb{R}$.

Uniform continuity of the objective functions

Let $(i, y, p, s) \in \{1, \ldots, m\} \times \{y_{\min}, \ldots, y_{\max}\} \times \frac{\delta}{2} \mathbb{N}^* \times \delta\{1, \ldots, K\}$ such that $p - \frac{s}{2} > 0$.

The function $(t, x) \rightarrow v_i(t, x, y, p, s)$ is uniformly continuous on $[0, T] \times [x_{\min}, +\infty)$. 
Dynamic programming principle

Let \((i, t, z) := (i, t, x, y, p, s) \in \{1, \ldots, m\} \times S\). Let \(\nu\) be a stopping time in \(\mathcal{T}_t, T\), we have

\[
v_i(t, z) = \sup_{\alpha \in \mathcal{A}(t, z)} \mathbb{E} \left[ v_{\nu \wedge \hat{\theta}} \left( \nu \wedge \hat{\theta}, Z_{\nu \wedge \hat{\theta}}^{t, i, z, \alpha} \right) \mathbf{1}_{\{\nu \wedge \hat{\theta} < \hat{\tau}_\alpha\}} \right.
\]

\[+ U_L \left( \hat{\tau}_\alpha, x e^{r(\hat{\tau}_\alpha - t)}, y, p, s \right) \mathbf{1}_{\{\hat{\tau}_\alpha \leq \nu \wedge \hat{\theta}\}} - g(y) \left( \nu \wedge \hat{\theta} \wedge \hat{\tau}_\alpha - t \right) \],

with \(\hat{\tau}_\alpha = \tau^{t, i, z, \alpha} \wedge T\) and

\[
\hat{\theta} = \inf\{u \geq t : N_u > N_{u^-} \text{ or } N_{u}^{a, i, t, z} > N_{u^-}^{a, i, t, z} \text{ or } N_{u}^{b, i, t, z} > N_{u^-}^{b, i, t, z}\}.
\]
Analytical properties of the objective function and dynamic programming principle

- Model and problem formulation
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- Numerical illustrations
HJB equation (1)

- **Set of admissible controls**: We define the following set:

\[
A(t, z) := \{ \alpha = (\varepsilon^a, \varepsilon^b, \eta^a, \eta^b) \in \{0, 1\}^4 : \delta \varepsilon^b < p - \frac{s}{2}, \\
\delta \leq s - \delta(\varepsilon^a - \varepsilon^b) \leq K\delta \text{ and } \delta \leq s + \delta(\eta^a - \eta^b) \leq K\delta \}.
\]

- **Transactions operators**: For all \((i, t, x, y, p, s) := (i, t, z) \in \{1, ..., m\} \times S\) and \(\alpha := \{\varepsilon^a, \varepsilon^b, \eta^a, \eta^b\} \in A(t, z)\), we introduce the two operators:

\[
\mathcal{A}\nu_i(t, z, \alpha) = \begin{cases} 
U_L(t, x, y_{min}, p, s) & \text{if } y = y_{min}, \\
v_i(t, x + p + \frac{s}{2}, y - 1, p + \frac{\delta}{2}(\eta^a + \eta^b), s + \delta(\eta^a - \eta^b)) & \text{else.}
\end{cases}
\]

\[
\mathcal{B}\nu_i(t, z, \alpha) = \begin{cases} 
U_L(t, x, y_{max}, p, s), & \text{if } y = y_{max} \\
U_L(t, z), & \text{if } x < x_{min} + p - \frac{s}{2} \text{ or } x = x_{min} + p - \frac{s}{2} < 0 \\
v_i(t, x - p + \frac{s}{2}, y + 1, p - \frac{\delta}{2}(\varepsilon^a + \varepsilon^b), s - \delta(\varepsilon^a - \varepsilon^b)) & \text{else.}
\end{cases}
\]
Let \((\varphi_i)_{1 \leq i \leq m}\) a family of smooth functions defined on \(S\). We introduce the following operator associated with state \(i \in \{1, .., m\}\):

\[
\mathcal{H}_i(t, z, \varphi_i, \frac{\partial \varphi_i}{\partial x}) = rx \frac{\partial \varphi_i}{\partial x} + \sum_{j \neq i} \gamma_{ij} (\varphi_j(t, x, y, p, s) - \varphi_i(t, x, y, p, s)) - g(y) \\
+ \sup_{\alpha \in \mathcal{A}(t, z)} \left[ \lambda_i^a(t, p, s) (A\varphi_i(t, x, y, p, s, \alpha) - \varphi_i(t, x, y, p, s)) \\
+ \lambda_i^b(t, p, s) (B\varphi_i(t, x, y, p, s, \alpha) - \varphi_i(t, x, y, p, s)) \right] = 0.
\]

We consider the HJB equation:

\[
- \frac{\partial \varphi_i}{\partial t} - \mathcal{H}_i(t, z, \varphi_i, \frac{\partial \varphi_i}{\partial x}) = 0, \quad \text{for } (t, z) \in S, \quad (1)
\]

with the following boundary and terminal conditions:

\[
v_i(t, x_{\text{min}}, y, p, s) = U_L(t, x_{\text{min}}, y, p, s) \quad (2) \\
v_i(T, x, y, p, s) = U_L(T, x, y, p, s) \quad (3)
\]
Theorem:

The family of objective functions \((v_i)_{1 \leq i \leq m}\) is the unique family of functions such that

i) **Continuity condition**: For all 
\[(i, y, p, s) \in \{1, \ldots, m\} \times \{y_{\min}, \ldots, y_{\max}\} \times \delta \frac{\mathbb{N}}{2} \times \delta\{1, \ldots, K\}, (t, x) \rightarrow v_i(t, x, y, p, s)\]
is continuous on \(\{(t, x) \in [0, T) \times [x_{\min}, +\infty) : (t, x, y, p, s) \in S\}\}.

ii) **Growth condition**: There exists \(C_1, C_2\) and \(C_3\) positive constants such that
\[1 - C_1 - C_2 e^{C_3 p} \leq v_i(t, x, y, p, s) \leq 1, \quad \text{on } \{1, \ldots, m\} \times S.\]

iii) **Boundary conditions**: 
\[v_i(t, x_{\min}, y, p, s) = U_L(t, x_{\min}, y, p, s) \text{ and } v_i(T, x, y, p, s) = U_L(T, x, y, p, s).\]

iv) **Viscosity solution**: \((v_i)_{1 \leq i \leq m}\) is a viscosity solution of the system of variational inequalities (1) on \(\{1, \ldots, m\} \times S\).
Numerical illustrations

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Impact function:

\[ f(t, y) = \exp(0.09y(T - t)). \]

Intensity functions:

\[ \lambda^a_i(p, s) = \frac{\psi^a_i}{p} \exp(-s - 0.01(p - 1)) \quad \text{and} \quad \lambda^b_i(p, s) = \psi^b_i p \exp(-s + 0.01(p - 1)), \]

with \( \psi^a_1 = 120, \quad \psi^a_2 = 80, \quad \psi^b_1 = 80, \quad \psi^b_2 = 120. \)

Penalty and utility function:

\[ g(y) = y^2 \times 10^{-3} \quad \text{and} \quad U(l) = 1 - e^{-\gamma l}. \]
**Figure:** Optimal strategy when a sell market order arrives
Optimal strategy

**Figure:** Optimal strategy when a sell market order arrives
Bid and ask price paths
A stock holdings path

**Figure**: A stock holdings path
Controlled vs random strategies

\textbf{Figure:} Mean of 100 paths of $L(t, Z_t)$
Thank you!