Semimartingale properties of the lower Snell envelope in optimal stopping under model uncertainty

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Programa

Arbitrage prices in incomplete markets

Let X be a price process of a financial market and let $\mathcal{P}$ be its class of martingale measures.

Theorem

Let H *be the payoff process of an American option. Then, the set of arbitrage free prices is an interval with boundaries*

$$\pi_{\inf}(\mathrm{H}) := \inf_{P \in \mathcal{P}} \sup_{\tau \in \mathcal{T}} E_P[\mathrm{H}_\tau] \text{ and } \pi_{\sup}(\mathrm{H}) := \sup_{P \in \mathcal{P}} \sup_{\tau \in \mathcal{T}} E_P[\mathrm{H}_\tau].$$

See El Karoui and Quenez [2], Karatzas and Kou [6], Kramkov [7]...

Optimal exercise under model uncertainty

1. In decision theory, Ellsberg's paradox [3], highlights how the ambiguity about the distribution are crucial in understanding human decisions under risk and uncertainty.
   The solution to the paradox is given by the so-called maxmin preferences axiomatized by Gilboa and Schmeidler [5].
2. The axiomatic framework of Gilboa and Schmeidler [5] yields for each preference a family of probability measures under which utilities are quantified and the worst possible outcome is the utility assigned and under which decisions are taken.
3. In the setting of [5], time consistency in an intertemporal framework is axiomatized by Epstein and Schneider [4].

Assumptions

Definition

Let $\tau \in \mathcal{T}$ be a stopping time and Q_1 and Q_2 be probability measures equivalent to $\mathbb{P}$. The probability measure defined through

$$Q_3(A) := E_{Q_1}[Q_2[A \mid \mathcal{F}_\tau]], A \in \mathcal{F}_{\mathbb{T}}$$

is called the pasting of Q_1 and Q_2 in τ.

Assumption

The family $\mathcal{Q}$ of equivalent probability measures is stable under pasting.

Assumptions

Assumption

The process H *is a càdlàg positive* $\mathbb{F}$*-adapted process which is of class*(D) *with respect to each* $Q \in \mathcal{Q}$, *i.e.,*

$$\lim_{x\to\infty} \sup_{\tau\in\mathcal{T}} E_Q[\mathrm{H}_\tau; \mathrm{H}_\tau \geq x] = 0.$$

The stochastic process H *is upper semicontinuous in expectation from the left with respect to each probability measure* $Q \in \mathcal{Q}$. *That is, for any stopping time* θ *of the filtration* $\mathbb{F}$ *and an increasing sequence of stopping times* $\{\theta_i\}_{i\in\mathbf{N}}$ *converging to* θ, *we have*

$$\limsup_{i\to\infty} E_Q[\mathrm{H}_{\theta_i}] \leq E_Q[\mathrm{H}_\theta]. \tag{1}$$

Optimal stopping times

Theorem (Trevino [8])

Define

$$\tau_\rho^Q := \inf\{s \geq \rho \mid \mathrm{H}_s \geq U_s^Q\}. \tag{2}$$

Then, the random time

$$\tau_\rho^{\downarrow} := \text{ess inf}\left\{\tau_\rho^Q \mid Q \in \mathcal{Q}\right\}, \tag{3}$$

is a $\mathit{stopping\ time}$ *and it is optimal:*

$$\text{ess sup}_{\tau\in\mathcal{T}[\rho,\mathbb{T}]}\text{ess inf}_{Q\in\mathcal{Q}}E_Q\left[\mathrm{H}_\tau \mid \mathcal{F}_\rho\right] = \text{ess inf}_{Q\in\mathcal{Q}}E_Q\left[\mathrm{H}_{\tau_\rho^{\downarrow}} \mid \mathcal{F}_\rho\right]. \tag{4}$$

The lower Snell envelope is a $\mathcal{Q}$*-submartingale in stochastic intervals of the form* $[\rho, \tau_\rho^{\downarrow}]$.

The lower Snell envelope

Theorem (Trevino [9])

Under mild conditions, there exists an optional right-continuous stochastic process $U^{\downarrow} := \{U_t^{\downarrow}\}_{0 \leq t \leq \mathbb{T}}$ *such that for any stopping time* $\tau \in \mathcal{T}$

$$U_\tau^{\downarrow} = \text{ess inf}_{Q \in \mathcal{Q}} \text{ess sup}_{\rho \in \mathcal{T}[\tau, \mathbb{T}]} E_Q[\mathrm{H}_\rho \mid \mathcal{F}_\tau], \ \mathbb{P} - a.s.$$

- In a recent paper, Cheng and Riedel [1] investigate the robust stopping problem

$$U_\tau^{\downarrow} = \inf_{Q \in \mathcal{Q}} \sup_{\rho \in \mathcal{T}} E_Q[\mathrm{H}_\rho],$$

under g-expectations with backward differential stochastic equations techniques.

- Their solution consists in stopping as soon as the underlying process touches its lower Snell envelope. Moreover, they obtain a structural result which describes the lower Snell envelope as the sum of a process of bounded variation and a stochastic integral with respect to Brownian motion.

Assumption

There exists a probability measure $Q \in \mathcal{Q}$ *such that* H *is of the form*

$$\mathrm{H}_t = \mathrm{H}_0 + S_t^Q + L_t^Q - N_t^Q,$$

for S^Q *a* Q*-submartingale and* L^Q, N^Q *càdlàg non decreasing processes with* $S_0^Q = L_0^Q = N_0^Q = 0$*, and* $E_Q[N_{\mathbb{T}}^Q] < \infty$*.*

The lower Snell envelope for semimartingales

Theorem (Trevino [10])

Suppose the Assumptions 2.2 and 3.1 holds true. Define $V^Q := U^{\downarrow} + N^Q$. *Let* τ_1, τ_2 *be two stopping times with* $0 \leq \tau_1 \leq \tau_2 \leq \mathbb{T}$. *Then*

$$E_Q[V^Q_{\tau_1}] \leq E_Q[V^Q_{\tau_2}].$$

Thus, V^Q *is a* Q*-submartingale.*

sketch of the proof

For $\delta > 0$ we set

$$\theta_\delta^{(1)} := (\tau_1 + \delta) \wedge \tau_2$$
$$\theta_\delta^{(2)} := \tau^{\downarrow}_{\theta_\delta^{(1)}} \wedge \tau_2.$$

Now for $i > 2$ we define recursively

$$\theta_\delta^{(i)} := (\theta_\delta^{(i-1)} + \delta) \wedge \tau_2, \text{ for } i \text{ odd},$$
$$\theta_\delta^{(i+1)} := \tau^{\downarrow}_{\theta_\delta^{(i)}} \wedge \tau_2, \text{ for } i+1 \text{ even}.$$

For $N > \frac{\mathbb{T}}{\delta}$ we have

$$\sum_{i=1}^{N} E_Q \left[V^Q_{\theta_\delta^{(i+1)}} - V^Q_{\theta_\delta^{(i)}} \right] = \sum_{i=1}^{\infty} E_Q \left[V^Q_{\theta_\delta^{(i+1)}} - V^Q_{\theta_\delta^{(i)}} \right].$$

sketch of the proof, cont.

For $i+1$ even we have

$$E_Q\left[V^Q_{\theta_\delta^{(i+1)}} - V^Q_{\theta_\delta^{(i)}}\right] \geq 0, \tag{5}$$

due to the $\mathcal{Q}$-submartingale property of Theorem 3.
For $i+2$ odd we show that

$$V^Q_{\theta_\delta^{(i+2)}} - V^Q_{\theta_\delta^{(i+1)}} \geq S^Q_{(\tau^{\downarrow}_{\theta_\delta^{(i)}}+\delta)\wedge\tau_2} - S^Q_{\tau^{\downarrow}_{\theta_\delta^{(i)}}\wedge\tau_2} + L^Q_{(\tau^{\downarrow}_{\theta_\delta^{(i)}}+\delta)\wedge\tau_2} - L^Q_{\tau^{\downarrow}_{\theta_\delta^{(i)}}\wedge\tau_2}. \tag{6}$$

Assumptions

There is a "uniform-version" of Theorem 5 under the stronger condition of the next

Assumption

There exists a $\mathcal{Q}$*-submartingale* $\{S_t\}_{0\le t\le \mathbb{T}}$ *with* $S_0 = 0$ *such that* H *is of the form*

$$\mathrm{H}_t = \mathrm{H}_0 + S_t + L_t - N_t,$$

for L, N *càdlàg non decreasing processes with* $L_0 = N_0 = 0$, *and*

$$E_Q[N_{\mathbb{T}}] < \infty,$$

for each $Q \in \mathcal{Q}$.

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Uniform decomposition of the lower Snell envelope

Theorem

Assume H *satisfies the Assumptions 2.2 and 3.2. Then* $V := U^{\downarrow} + N$ *is a* $\mathcal{Q}$*-submartingale.*

Let B be a Brownian motion defined in the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ in this section we assume that $\mathbb{F}$ is the augmented filtration generated by B.

Definition

The class $\mathbb{W}$ consists of probability measures Q^{γ} determined by density processes satisfying the Dooleans-Dade stochastic equation

$$dZ_t = \gamma_t Z_t dB_t, \tag{7}$$

with γ a progressively measurable process satisfying

1. $\gamma_t \leq 0$ for $t \in [0, \mathbb{T}]$, and
2. $E\left[\int_0^{\mathbb{T}} \gamma_s^2 ds\right] < \infty$.

Theorem

Take a non negative continuous deterministic function V defined in the interval $[0, \mathbb{T}]$. Let $\mathcal{H} = B + V$. Then, the lower Snell envelope of the process $\mathcal{H}$ is equal to $\mathcal{H}$ itself.

Theorem

There exists no continuous uniform decomposition as the difference of a $\mathcal{Q}$-submartingale and a non decreasing process for the Brownian motion B with respect to $\mathbb{W}$.

Theorem

Let U be a non negative $\mathbf{E}^{\downarrow}$-supermartingale of class(D) with respect to each element Q of $\mathcal{Q}$. Let Ξ be the class of finite partitions with points in D, a countable dense subset of $[0,\mathbb{T}]$. Let $DM(\Xi)$ be the class of random variables defined by

$$V_{\mathbb{T}}^{\Pi} = \sum_{i=0}^{n-1} \left(U_{t_i} - \mathbf{E}^{\downarrow}[U_{t_{i+1}} \mid \mathcal{F}_{t_i}] \right), \tag{8}$$

for $\Pi = \{0 = t_0 < t_1 < \ldots < t_n = \mathbb{T}\}$ a partition in Ξ.
If the class $DM(\Xi)$ is uniformly integrable with respect to a $Q_0 \in \mathcal{Q}$, then there exists a right-continuous non decreasing process $\mathcal{V}$ such that $\mathcal{Z} := U + \mathcal{V}$ is a Q_0-submartingale. In this case, for each probability measure Q under which the class $DM(\Xi)$ is uniformly integrable, the process $\mathcal{Z}$ is also a submartingale.

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