Principal-Agent Problem and Dynamic Programming

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Outline

Output process

- Effort = Control process $\nu = (\alpha, \beta)$ with values in $A \times B$.

- Output = the controlled state process in $\mathbb{R}^d$: any weak solution $\mathbb{P}$ of the (non-Markovian) SDE

$$dX_t = \sigma_t(X, \beta_t)\big[\lambda_t(X, \alpha_t)dt + dW_t\big].$$

where W is a Brownian motion with values in $\mathbb{R}^n$.

- Observation of X does not give access to the drift $\sigma\lambda$.

- Observation of X gives access to $\sigma\sigma^\top$ but not to σ.

The Agent problem

Agent solves the following control problem :

$$V_0^A(\xi) := \sup_{\mathbb{P}} \mathbb{E}^{\mathbb{P}}\Bigg[\underbrace{e^{-\int_0^T k_s(X,\nu_s^{\mathbb{P}})ds}}_{Discount}\underbrace{\xi(X)}_{Salary} - \int_0^T e^{-\int_0^t k_s(X,\nu_s^{\mathbb{P}})ds}\underbrace{c_t(X,\nu_t^{\mathbb{P}})}_{Cost}dt\Bigg],$$

where the contract $\xi(X)$ is $\mathcal{F}_T-$measurable, and represents the compensation for the management of X.

$\longrightarrow$ No interest on X, except for the compensation ξ indexed on X.

Path-dependency of ξ is crucial $\Longrightarrow$ Non-Markovian stochastic control

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The Principal problem

Moral hazard : Principal chooses the optimal compensation scheme $\xi(X)$ based on the observation of X only, i.e. Principal does not observe the Agent effort.

Principal solves the optimization problem

$$V_0^P := \sup_{\xi \in \Xi_R} \mathbb{E}^{\mathbb{P}^\star(\xi)} \Big[U(\ell(X_T) - \xi) \Big]$$

- $\ell(X_T)$: liquidation value of the output.
- $\mathbb{P}^\star(\xi)$: optimal solution of Agent problem given the contract ξ.
- Ξ_R : collection of all ξ, such that $V_0^A(\xi) \geq R$ (reservation utility, participation constraint).

Existing literature

- Non-zero sum Stackelberg game, highly nonlinear problem ;
- Mainly static or discrete time problems : *Spear & Srivastava*, *Salanié*, *Tirole*, *Laffont*, *Martimort*, *Radner*, *Jofre*,. . . $\Longrightarrow$ very general but limited computations ;
- Extension to continuous time models : *Holmström & Milgrom*, *Schättler*, *Williams*, *Sung*, *Sannikov*, *Cvitanić & Zhang*, *Ekeland*,... ;
- Main method for general problems : calculus of variations $\Longrightarrow$ Pontryagin Maximum Principle leading to a system of Forward-Backward SDEs...
- **Our objective :** Simple solution by standard dynamic programming.

Outline

Stochastic control of Markov diffusions

Probability space $(\Omega, \mathcal{F}, \mathbb{P})$, filtration $\mathbb{F}$, $W : \mathbb{R}^n-$Brownian motion.

- Control process : $\nu = \{\nu_t, t \geq 0\}$ $\mathbb{F}-$progressively measurable process with values in $U \subset \mathbb{R}^k$.
- Controlled state process X^ν, valued in $\mathbb{R}^d$, defined by the SDE

$$dX_t^\nu = b(t, X_t^\nu, \nu_t)dt + \sigma(t, X_t^\nu, \nu_t)dW_t.$$

- $\mathcal{U}$: admissible controls, i.e. X^ν well-defined, appropriate regularity.
- Control problem :

$$V(t,x) := \sup_{\nu \in \mathcal{U}} \mathbb{E}\left[g(X_T^{t,x,\nu})\right].$$

Hamiltonian and the HJB equation

- Hamiltonian :

$$H(t,x,z,\gamma) \quad := \quad \sup_{u\in U}\left\{b(t,x,u)\cdot z + \frac{1}{2}\sigma\sigma^\top(t,x,u):\gamma\right\},$$

for all $(t,x)\in[0,T]\times\mathbb{R}^d$ and $(z,\gamma)\in\mathbb{R}^d\times\mathcal{S}_\mathbb{R}(d)$. Then,

The value function V solves the Dynamic Programming (Hamilton-Jacobi-Bellman) Equation :

$$\begin{aligned}\partial_t V + H(t,x,DV,D^2V) = 0, &\quad t<T,\ x\in\mathbb{R}^d,\\ V(T,.) = g &\qquad \text{on } \mathbb{R}^d.\end{aligned}$$

In which sense does HJB equation hold ?

• Classical sense : $V \in C^{1,2}([0,T),\mathbb{R}^d)$... Not expected, many counter-examples.

• Sobolev solutions : $V \in W^{1,2}([0,T),\mathbb{R}^d)$ (Krylov 1980), very developed in the semilinear case (that is with uncontrolled diffusion)...

• Vicosity solutions (Crandall & Lions '81, Lions '83...) : No access to optimal control, in general.

To sum up...

- Easy extension to running costs and discounting.
- Gives a lot of hindsight into the corresponding non-Markovian control problems.
- Dynamic programming approach to characterize optimal controls (necessary for Principal problem !) $\longrightarrow$ need to have at least some weak notion of regularity.

Outline

The Agent's Problem

- Output now becomes

$$dX_t = \sigma_t(X)\big[\lambda_t(X,\alpha_t)dt + dW_t\big],$$

with associated weak solution $\mathbb{P}^\alpha$.

- Agent solves :

$$\sup_\alpha \mathbb{E}^{\mathbb{P}^\alpha}\left[e^{-\int_0^T k_s(X,\alpha_s^{\mathbb{P}^\alpha})ds}\xi(X) - \int_0^T e^{-\int_0^t k_s(X,\alpha_s^{\mathbb{P}^\alpha})ds}c_t(X,\alpha_t^{\mathbb{P}^\alpha})dt\right].$$

- The $\mathbb{P}^\alpha$ form a dominated family.

The Agent's Problem

- For any admissible α, let

$$V_t^{A,\alpha} := \mathbb{E}_t^{\mathbb{P}^\alpha}\left[e^{-\int_t^T k_s(X,\alpha_s^{\mathbb{P}^\alpha})ds}\xi(X) - \int_t^T e^{-\int_t^s k_r(X,\alpha_r^{\mathbb{P}^\alpha})dr} c_s(X,\alpha_s^{\mathbb{P}^\alpha})ds\right].$$

- Then

$$e^{-\int_0^t k_s(X,\alpha_s^{\mathbb{P}^\alpha})ds} V_t^{A,\alpha} - \int_0^t e^{-\int_0^s k_r(X,\alpha_r^{\mathbb{P}^\alpha})dr} c_s(X,\alpha_s^{\mathbb{P}^\alpha})ds,$$

is a $\mathbb{P}^\alpha-$martingale.

The Agent's Problem

• By martingale representation, there is some predictable Z^α, and some martingale M^α orthogonal to W with

$$V_t^{A,\alpha} = \xi(X) + \int_t^T f_s(X, V_s^{A,\alpha}, Z_s^\alpha, \alpha_s^{\mathbb{P}^\alpha})ds - \int_t^T Z_s^\alpha \cdot \sigma_s(X)dW_s - \int_t^T dM_s^\alpha,$$

where

$$f_t(x,y,z,\alpha) := -c_t(x,\alpha) - k_t(x,\alpha)y + \sigma_t(x)\lambda_t(x,\alpha) \cdot z.$$

• Linear Backward SDE (Pardoux & Peng '91, El Karoui, Peng & Quenez '97, El Karoui & Huang '97...) !

The Agent's Problem

- Using the ideas of Hamadène and Lepeltier ('95) and El Karoui and Quenez ('95), by comparison theorem for BSDEs, we have

$$V_t^A = \xi(X) + \int_t^T F_s(X, V_s^A, Z_s)ds - \int_t^T Z_s \cdot \sigma_s(X) dW_s - \int_t^T dM_s,$$

where

$$F_t(x,y,z) := \sup_\alpha f_t(x,y,z,\alpha),$$

is the Hamiltonian associated to the problem of the Agent !

- If F is Lipschitz and ξ and $F_t(x,0,0) \in L^p$, then wellposedness is ensured.

The Agent's Problem : conclusion

- Optimal control of Agent is given by any $\mathbb{P}^{\alpha^\star}$ where $\alpha^\star$ attains the sup in F.

- This requires only integrability assumptions on ξ, λ and σ.

The case of controlled diffusion : difficulties

• Similar to the Markov case, very difficult to access to the Hessian component... Need a relaxation of the C^2-regularity.

• The $\mathbb{P}^\nu$'s (measures induces by the controlled state) are defined on different supports for different values of ν, so can not reduce the analysis to one single measure.

$\Longrightarrow$ Quasi-sure stochastic analysis : stochastic analysis under a non-dominated family of singular measures.

Outline

From fully nonlinear HJB equation to semilinear

- $H_t(x,y,z,\gamma)$ non-decreasing and convex in γ, Then

$$H_t(x,y,z,\gamma) \quad = \quad \sup_{a\geq 0}\left\{\frac{1}{2}a:\gamma - H_t^*(\omega,z,a)\right\}.$$

- HJB equation is

$$\partial_t v + \sup_{a\geq 0}\left\{\frac{1}{2}a:\partial^2_{xx}v - H_t^*(x,v,\partial_x v,a)\right\} \quad = \quad 0.$$

- Stochastic representation

$$v_t \quad = \quad \sup_a \; Y_t^a,$$

where,

$$Y_t^a = \xi - \int_t^T H_s^*(Y_s^a, Z_s^a, a_s)ds + \int_t^T Z_s^a dX_s, \quad \mathbb{P}^a - \text{a.s.}$$

Wellposedness of second order BSDEs

There exists a unique triple (Y, Z, K) $\mathbb{F}-$adapted with appropriate integrability, such that

• $Y_t = \xi - \int_t^T H_s^*(Z_s, a_s)ds - \int_t^T Z_s dX_s + \int_t^T dK_s$, $\mathbb{P}^a-$a.s. , for all control process a,

• K nondecreasing, $K_0 = 0$, and $\inf_a \mathbb{E}^{\mathbb{P}^a}\left[K_T\right] = 0$.

• Cheredito, Victoir, Soner, Touzi, Zhang, Matoussi, Neufeld, Nutz, P., Tan, Zhou.

Regularity reduces to the non-decreasing process K

Suppose $K_t = \int_0^t \dot{K}_s ds,\ t \in [0,T]$, and define the process Γ by

$$\dot{K}_t = H_t(Y_t, Z_t, \Gamma_t) - \frac{1}{2} a_t : \Gamma_t + H_t^*(Y_t, Z_t, a_t),$$

Substituting in the 2BSDE, we get for all a :

$$Y_t = \xi + \int_t^T \Big[H_s(Y_s, Z_s, \Gamma_s) - \frac{1}{2} a_s : \Gamma_s \Big] ds - \int_t^T Z_s dX_s,\ \mathbb{P}^a - \text{a.s.}$$

$\Longrightarrow$ Use this as natural class of contracts !

A class of revealing contracts

- For $Y_0 \in \mathbb{R}$ and Z, Γ $\mathbb{F}^X-$prog meas, define

$$Y_t^{Z,\Gamma} = Y_0 + \int_0^t Z_s \cdot dX_s + \frac{1}{2}\int_0^t \Gamma_s : d\langle X \rangle_s - \int_0^t H_s(X, Y_s^{Z,\Gamma}, Z_s, \Gamma_s) ds$$

Proposition $V_0^A(Y_T^{Z,\Gamma}) = Y_0$ and any maximizer of the Hamiltonian $(a^\star, b^\star)(Y, Z, \Gamma)$ induces a solution $\mathbb{P}^{\star, Z, \Gamma}$ of the Agent problem .

Sub-optimal stochastic control problem

- Under $\mathbb{P}^{\star,Z,\Gamma}$, we have

$$\begin{aligned} dX_t &= \sigma_t^\star(X, Y_t, Z_t, \Gamma_t)\big[\lambda_t^\star(X, Y_t, Z_t, \Gamma_t)dt + dW_t\big], \\ dY_t^{Z,\Gamma} &= Z_t \cdot dX_t + \frac{1}{2}\Gamma_t : d\langle X\rangle_t - H_t(X, Y_t^{Z,\Gamma}, Z_t, \Gamma_t)dt, \end{aligned}$$

where $\sigma_t^\star(\omega, y, z, \gamma) := \sigma_t\big(\omega, b^\star(\omega, y, z, \gamma)\big)$, $\lambda_t^\star(\omega, y, z, \gamma) := \cdots$

$$V_0^P \geq \sup_{Y_0 \geq R} \underline{V}_0(X_0, Y_0); \;\; \underline{V}_0(X_0, Y_0) := \sup_{Z,\Gamma} \mathbb{E}^{\mathbb{P}^{\star,Z,\Gamma}}\Big[U(\ell(X_T) - Y_T^{Z,\Gamma})\Big]$$

- $\underline{V}$ characterized by standard HJB equation

The general case

• In the fully nonlinear case, the representation $\xi = Y_T^{Z,\Gamma}$ not true for general ξ... We only have the 2BSDE representation :

$$Y_t = \xi - \int_t^T H_s^*(Y_s, Z_s, a_s)ds - \int_t^T Z_s dX_s + \int_t^T dK_s, \quad \mathbb{P} - \text{a.s. for all } \mathbb{P},$$

with K nondecreasing, $K_0 = 0$, and $\inf_{\mathbb{P}} \mathbb{E}^{\mathbb{P}}\left[K_T\right] = 0$

• It is sufficient to find an approximation ξ^ε of ξ such that $\xi^\varepsilon = Y_T^{Z^\varepsilon,\Gamma^\varepsilon}$... and pass to the limit in the Principal problem...

The general case

- $K_t^\varepsilon := \frac{1}{\varepsilon}\int_{0\vee(t-\varepsilon)}^t K_s ds$ and $\xi^\varepsilon := Y_T^\varepsilon$ (replacing K by K^ε) $\Longrightarrow$ $\xi^\varepsilon = Y^{Z,\Gamma^\varepsilon}$ and $\mathbb{P}^\star$ optimal for K is also optimal for K^ε.

$$\Longrightarrow \quad V_0^P \quad = \quad \sup_{Y_0 \geq R} \underline{V}_0(X_0; Y_0).$$

Thank you for your attention !