Stochastic control for insurance: new problems and methods

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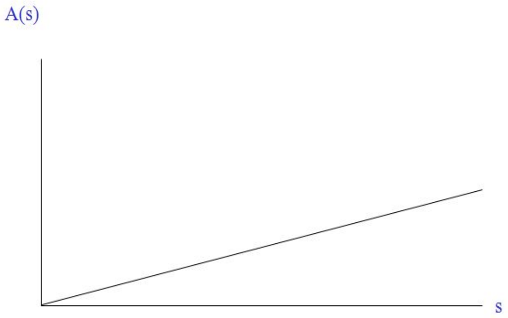
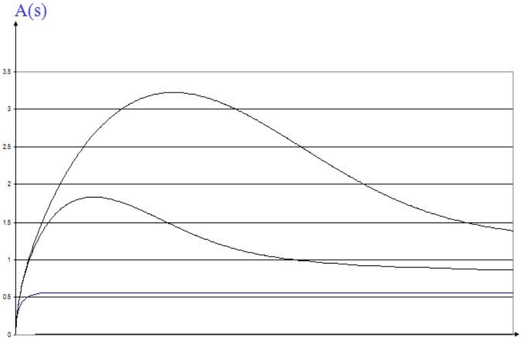
Naive investment

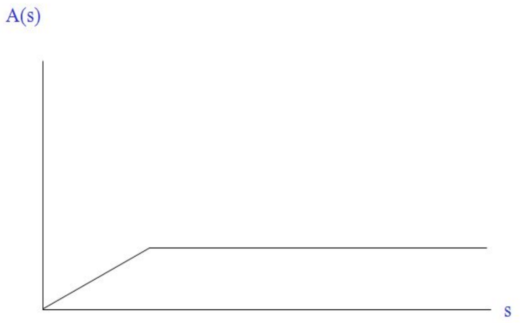


Browne (1995) investment

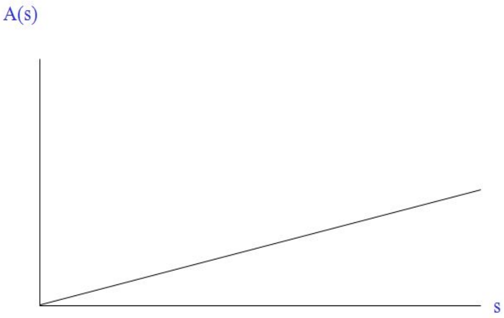
H + Plum (2000) investment



Small claims investment



Large claims investment



Stochastic control in Finance

Stochastic control **in Finance** started more than 40 years ago (1969/1971) with Robert Merton's papers

"Lifetime Portfolio Selection under Uncertainty: the Continuous-Time Case", The Review of Economics and Statistics 1969 and

"Optimum Consumption and Portfolio Rules in a Continuous-Time Model", Journal of Economic Theory, 1971.

Option pricing formula

giving rise to the famous option pricing articles by

Robert Merton: "Theory of Rational Option Pricing". Bell Journal of Economics and Management Science, 1973 and

Fischer Black and Myron Scholes: "The Pricing of Options and Corporate Liabilities". Journal of Political Economy 1973.

Standard textbooks

Wendell Fleming and Raymond Rishel: Deterministic and Stochastic Optimal Control, 1975

Wendell Fleming and Mete Soner: Controlled Markov Processes and Viscosity Solutions, 2006

Robert Merton: Continuous Finance, 1990

Ioannis Karatzas and Steven Shreve: Methods of Mathematical Finance, 1998

and the work of Marc Yor, Bernt Øksendal, Søren Asmussen, and Jerome Stein.

Stochastic Control for Insurance 1967

Karl Borch (NHH Bergen, Norway), Royal Statistical Society of London, 1967:

The theory of control processes seems to be *taylor made* for the problems which actuaries have struggled to formulate for more than a century. It may be interesting and useful to meditate a little how the theory would have developed if actuaries and engineers had realized that they were studying the same problems and joined forces over 50 years ago. A little reflection should teach us that a *highly specialized* problem may, when given the proper mathematical formulation, be identical to a series of other, seemingly unrelated problems.

Stochastic Control in Insurance since 1995

Sid Browne: Optimal investment policies for a firm with a random risk process: exponential utility and minimizing the probability of ruin, 1995

Hanspeter Schmidli: Stochastic Control in Insurance, 2008 (References)

Huyen Pham: Continuous-time Stochastic Control and Optimization with Financial Applications, 2009

Pablo Azcue and Nora Muler: Stochastic Optimization in Insurance: A Dynamic Programming Approach, 2014

Latest work

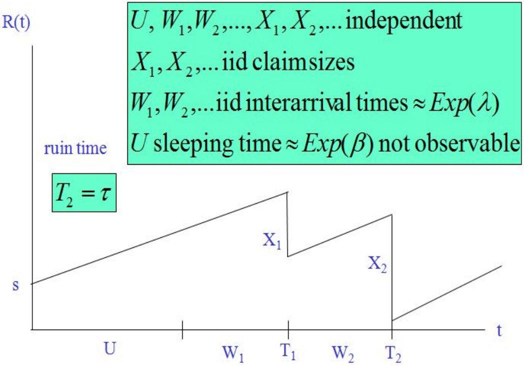
Optimal dividend and reisurance strategies with financing and liquidation value.

Dingjun Yao, Hailjang Yang, Ronming Wang

ASTIN Bulletin 46 (2), pp. 365-399, 2016.

A 1st introductory example: The sleeping volcano

Volcanos show long waiting times between periods with frequent seismic waves. One could model claims caused by these waves as a delayed Lundberg process:



A 1st introductory example: The sleeping volcano

What is the optimal reinsurance strategy to minimize ruin probability?

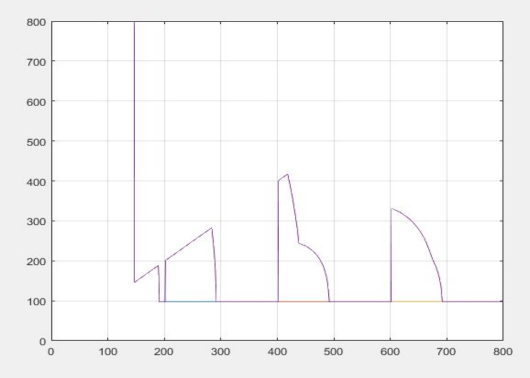
Clearly, no reinsurance at the beginning. But when should we start reinsuring? Too early=waste money for nothing. Too late: full first claim to be paid.

After the first claim: no longer sleeping, optimal reinsurance in the Lundberg model.

Cox model: suited also for model uncertainty.

Solution with XL Re: $X = \min(X, M) + (X - M)^+$

For different initial surplus s we obtain different optimal strategies for the time up to the first claim. Results are shown for $s = 100,\ 200,\ 400$ and 600. Safety first strategies!

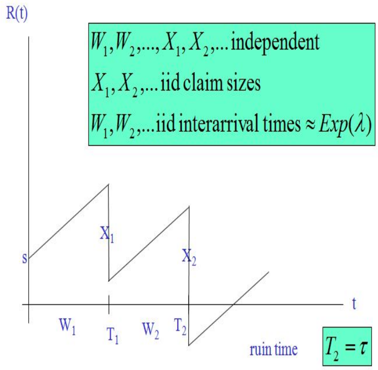


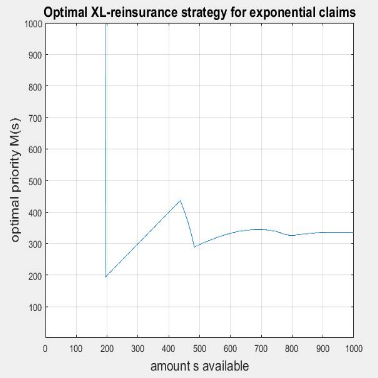
New: Dividend payment under ruin constraints

- Value of the company: expected discounted sum of dividends (de Finetti 1957).
- Optimal value: leads to certain ruin. And vice versa.
- Two objectives problem.
- Classical methods not applicable: not for solutions nor for numerical results.
- Constraints are cheap: little reduction of firm value.
- Lundberg case: form of the optimal strategy is known, enables computation.

Optimal XL reinsurance for Lundberg models

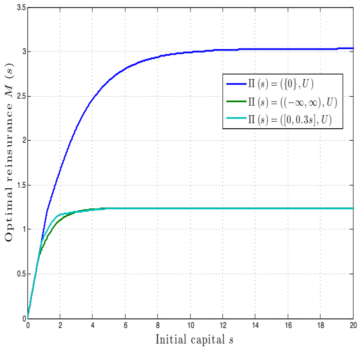
unlimited excess of Loss reinsurance with priority M: for claim X the reinsurer pays $(X - M)^+$. Exp claims and optimal strategy $M(s)$.

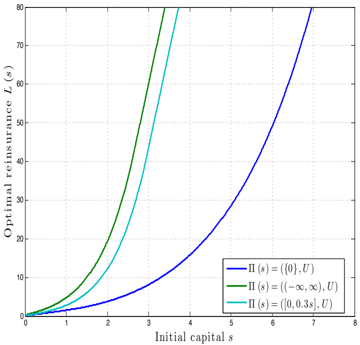




Limited XL for Pareto claims

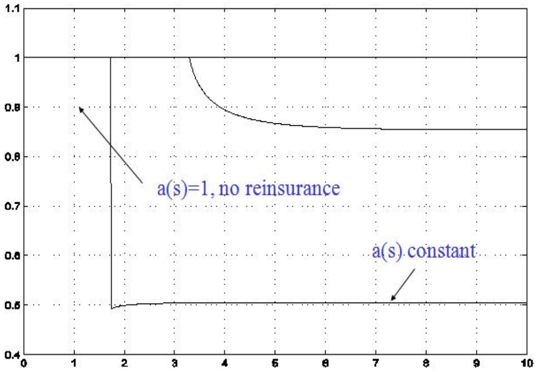
Limited XL: reinsurer pays $\min[L, (X - M)^+]$. Reinsurance starts at $s = 0$.



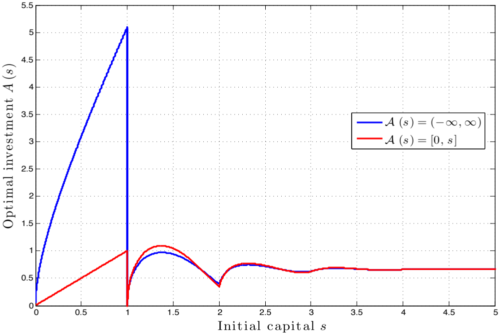


Proportional reinsurance for Lundberg models

proportional reinsurance with quota $0 < a < 1$: for claim X the reinsurer pays $(1 - a)X$. Exp claims <83 cases) and optimal strategy $a(s)$.

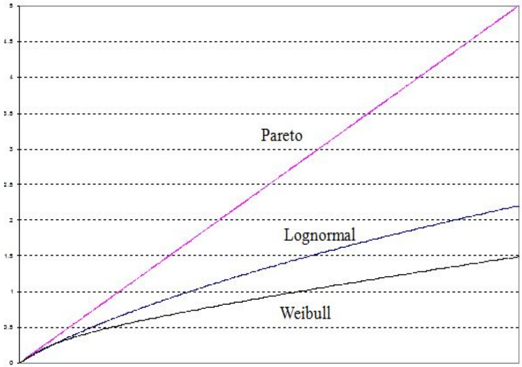


Optimal investment, (un-)constrained, $X = 1$



Unconstrained optimal investment, large claims

The more risk in insurance claims, the more market risk you take.



Unconstrained optimal investment

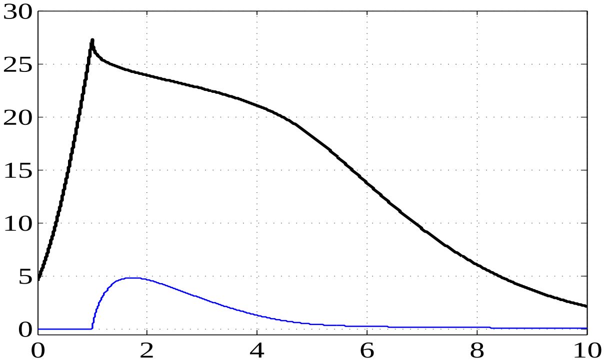
- Small claims: fast convergence to the constant $1/R$, where R is the adjustment coefficient of the problem.
- Large claims: $A(s) \to \infty$, known speed.
- Always: $A(s) \sim C\sqrt{s}$, unlimited leverage, not admissible.
- Numerically easy: Plum and H (2003).

Constrained optimal investment

Reasonable constraints:

- $A(s) \leq s$ no leverage.
- $A(s) \geq 0$ no short selling.
- $as \leq A(s) \leq bs$ bounded leverage and short selling.
- $A(s) = 0$ for small s, $A(s) \geq 0$ for large s.

Upward jump in the constraints, Exp, $V'(s), A(s)/s$



$$\mathcal{A}(s) = \{0\}, s < 1, \mathcal{A}(s) = [0, \infty), s \geq 1$$

Downward jump in the constraints, Exp, $V'(s), A(s)/s$

$$\mathcal{A}(s) = [0, \infty), s < 1, \mathcal{A}(s) = \{0\}, s \geq 1$$

Consequence for the Hamilton-Jacobi-Bellman 2nd order integro differential equation

The dynamic equation for the mentioned control problem, valid for $s > 0$, is

$$0 = \sup_{A \in \mathcal{A}(s)} \left\{ \lambda E[V(s-U) - V(s)] + (c+A)V'(s) + A^2 V''(s)/2 \right\}.$$

No hope for the statement: the value function is the unique smooth solution of the above HJB.

Instead: the value function is the unique (smooth) viscosity solution of the above HJB.

Computation for optimal investment

Euler type discretisation works in most cases!

- Replace the expectation $E[V(s-U)]$ by a sum with total mass 1:
$$\sum_i V(s-i\Delta)\mathbb{P}\{i\Delta \leq U < i\Delta\}.$$
- Replace $V'(s)$ by $(V(s+\Delta)-V(s))/\Delta$.
- Replace $V''(s)$ by $(V'(s)-V'(s-\Delta))/\Delta$.

Convergence can be proven in the Fleming-Soner viscosity style.

Optimal dividend payment under a ruin constraint

Solution in the Lundberg model with exp claims (mean 1): properties of the optimal strategy for fixed initial surplus s and allowed ruin probability α :

- In each epoch between claims we have a constant barrier $M(z)$ depending only on the state Z after the claim.
- With the adjustment coefficient $R = 1 - \lambda/c$ and discount rate δ we have

$$\begin{aligned} M(Z) &= H_n + \rho Z, \\ \rho &= (R-\gamma)/(1-\gamma), \\ \gamma &= (\lambda+\delta)/c. \end{aligned}$$

- ρ is negative when $\gamma < 1$ and $c < 2\lambda + \delta$.
- $\rho = 0$ when $c = 2\lambda + \delta$.

A heuristic improvement procedure H (2016)

For a suboptimal dividend function $V_n(s,\alpha)$ and for $U \geq s$ and $\alpha > \psi(s)$ define $a(U)$ as the solution to

$$\alpha = \frac{\psi(s)-\psi(U)}{1-\psi(U)} + a(U)\frac{1-\psi(s)}{1-\psi(U)}. \tag{1}$$

Then a better suboptimal value function $V_{n+1}(s,\alpha)$ is given by

$$\begin{aligned} G(s,\alpha) &= \max_{U\geq s} W(s)V_n(U,a(U))/W(U), \\ V_{n+1}(s,\alpha) &= \max(G(s,\alpha), V_{n+1}(s-1,\alpha)+1), \end{aligned}$$

where the second maximum is taken only if $\psi(s-1) \leq \alpha$.

Justification of the heuristics

- Wait until you reach $U \geq s$, without paying dividends.
- You will be ruined before reaching U with probability

$$q(U) := (\psi(s) - \psi(U))/(1 - \psi(U).$$

- You will reach U before ruin with probability

$$(1 - \psi(s))/(1 - \psi(U)) = 1 - q(U).$$

- When reaching U you can pay the dividend value $V(U, a(U))$ discounted by $E[exp(-\delta\tau)] = W(s)/W(U)$ with τ the waiting time until you reach U from s.
- $x \to W(x))$ is a solution of the dynamic equation for dividends without constraint.

Properties of the procedure

- Convergence not obvious. Monotone convergence to a suboptimal solution.
- Applicable for many models and for other control problems with simple action space.
- No Hamilton-Jacobi-Bellman equation needed.
- In each step, a bivariate grid of values has to be computed.

Viscosity solutions

Viscosity solutions are used in cases in which value functions are not (known to be) smooth. There a standard method is the comparison argument: Two viscosity solutions are $\leq$ on $[0, \infty)$ when they are $\leq$ at the two points 0 and ∞. In finance value functions have fixed values at 0 and ∞, but this is no longer true in insurance: e.g. ruin probabilities $\psi(s)$ are 1 at ∞, but unknown and positive at 0.

A second boundary value is for the derivative at 0, e. g. for ruin probability

$$\psi'(0) = \lambda/c.$$

Comparison argument with derivative

In H (2015) it is shown that under weak assumptions two viscosity solution of an order two integro differential HJB-equation which have continuous derivatives are $\leq$ when they are $\leq$ at 0 and their derivatives are $\leq$ at 0.

Used in the proof for convergence of Euler schemes in the optimal investment problem.

The use of non stationary models: Dividend problem

In discrete time $0, 1, 2, ...$ $d(i)$ is the dividend paid at time i which is discounted by r^i. For a stationary Markov surplus process $S(i), i = 0, 1, 2, ...$ we want to maximize the dividend value

$$J^d(s) = E[\sum_{i=1}^{\infty} r^i d(i) | S(0) = s],$$

subject to the constraint for the ruin probability

$$\psi^d(s) = \mathbb{P}\{S(i) - d(1) - ... - d(i) < 0 \text{ for some } i > 0\} \leq \alpha.$$

We use the Lagrange multiplier method and maximize instead

$$V^d(s) = J^d(s) - L\,\psi^d(s).$$

Dividend problem with Lagrange: a fast algorithm

Taksar algorithm: Consider the time dependent values

$$J^d(s,t) = E[\sum_{i=1}^{\infty} r^{t+i} d(i) | S(0) = s],$$

which is the dividend value after time t, discounted to time 0 and the probabilities $\psi^d(s,t)$ for ruin after time t as well as the Lagrange functions

$$V^d(s,t) = J^d(s,t) - L\,\psi^d(s,t).$$

Advantage: discounting is not in the dynamic equation, and we can use a simple dividend strategy (no dividends) for large T.

Dynamic equation in the de Finetti model

$$
\begin{aligned}
V(s,T) &= -L\,\psi^0(s),\\
G(s,t) &= pV(s+1,t+1)+qV(s-1,t+1)\\
V(s,t) &= \max[G(s,t),V(s-1,t)+r^t],\\
V(s) &= V(s,0).
\end{aligned}
$$

Similar setup in other stationary Markov models.
Mind the Lagrange gap!

The use of non stationary models: Cox models

Cox processes are Lundberg processes with a random (unobservable) intensity. The intensity is a finite state homogeneous Markov process. The model is Markov with respect to the two variables $s(t)=$surplus and the conditional distribution $p(t)$ for the intensity at time t, given the observation until t.

Between claims, $p(t)$ satisfies a first order differential equation (in x!). This reduces the dimension of the problem (between jumps) by one.

Compare our example 1 in which computation could be done for each s separately.

Conclusion

New problems:

- Constrained optimal investment.
- Optimal dividend payment with ruin constraints.
- Optimal control for Cox processes.

New methods:

- Euler type discretisations
- Non stationary approaches
- Univariate approaches to seemingly bivariate problems.

Some references not in Schmidli's book

- Azcue P, Muler N (2014) Stochastic Optimization in Insurance: A Dynamic Programming Approach. Springer.
- Edalati A (2013) Optimal constrained investment and reinsurance in Lundberg insurance models. Thesis, KIT.
- Edalati A, Hipp C (2014) Solving a Hamilton-Jacobi-Bellman equation with constraints. Stochastics.
- Hernandez C, Junca M (2015) Optimal dividend payments under a time of ruin constraint: Exp claims. IME.
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- Pham H (2009) Continuous-time Stochastic Control and Optimization with Financial Applications. Springer.

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Thank you!