UNIVERSIDADE DA CORUÑA

PDE models for pricing fixed rate mortgages and their insurance and coinsurance

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Outline

1. Fixed rate mortgage contract. Objectives
2. Mathematical models
   - PDE model
   - PIDE model
   - Additional mortgage characteristics
3. Numerical methods for PDE and PIDE models
4. Numerical results
5. Conclusions

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Mortgage contracts

A **mortgage** is a financial contract in which the borrower obtains funds usually from a bank or a financial institution using the house as a collateral.

Some characteristics of the contract:

- Interest rate:
  - **Fixed-Rate Mortgages (FRM)**.
  - Adjustable-Rate Mortgages (ARM).
- The term of the loan.
- Amount and frequency of payments: monthly payments.
- Prepayment option for the borrower under a penalization cost.
- Default option: insurance on the loan seems suitable.

Fixed rates does not mean this...

"You have been approved for a fixed-rate mortgage. That means if interest rates go up again and you're not paying enough, we'll fix it."

Aim of the work

- Pricing of a mortgage with prepayment and default options.
- Obtain the value of the insurance and the coinsurance.
- Determine the fixed interest rate as an equilibrium rate.
- Prepayment can occur at any time during the life of the loan.
- Default happens only at monthly payment dates.
- PDE modelling approach: one PDE problem for each month.
- Prepayment option as a free boundary problem (one for each month): optimal prepayment boundary.
- Jump-diffusion models for the house value: PIDE problem.
- Design appropriate numerical methods for solving the models.

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Underlying stochastic factors (I)

Evolution of the house value: geometric Brownian motion

$$dH_t = (\mu - \delta)H_t dt + \sigma_H H_t dX_t^H$$

where

- μ : the instantaneous average rate of house-price appreciation,
- δ : the 'dividend-type' per unit service flow provided by the house,
- σ_H : the house-price volatility
- X_t^H : the standardized Wiener process for house price.

Equivalently, under the risk neutral measure the process satisfies

$$dH_t = (r_t - \delta)H_t dt + \sigma_H H_t dX_t^H$$

Underlying stochastic factors (II)

Evolution of interest rate: CIR mean-reverting square root process

$$dr_t = \kappa(\theta - r_t)dt + \sigma_r \sqrt{r_t} dX_t^r$$

where

- κ : the speed of adjustment in the mean reverting process,
- θ : the long term mean of the short-term interest rate r_t,
- σ_r : the interest-rate volatility
- X_t^r : the standardized Wiener process for interest rate.

Correlated Wiener processes

$$dX_t^H dX_t^r = \rho dt$$

Dynamics of any asset (derivative) depending on house prices and interest rate

- $F_t = F(t, H_t, r_t)$: price of any asset whose value depends on the house price H_t, interest rate r_t and time t.
- Applying Itô's Lemma and suppressing the dependence on t for simplicity, the dynamics of F_t satisfies:

$$\begin{aligned} dF &= \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial H}dH + \frac{\partial F}{\partial r}dr \\ &\quad + \frac{1}{2}\left(\sigma_H^2 H^2 \frac{\partial^2 F}{\partial H^2} + 2\rho\sigma_H\sigma_r H\sqrt{r}\frac{\partial^2 F}{\partial H \partial r} + \sigma_r^2 r \frac{\partial^2 F}{\partial r^2}\right) dt \end{aligned}$$

PDE for any asset depending on house price and interest rate

Let $\Omega = (0, +\infty)^2$

PDE in $(0, T) \times \Omega$

$$\frac{\partial F}{\partial t} + \frac{1}{2}\sigma_H^2 H^2 \frac{\partial^2 F}{\partial H^2} + \rho\sigma_H\sigma_r H\sqrt{r}\frac{\partial^2 F}{\partial H \partial r} + \frac{1}{2}\sigma_r^2 r \frac{\partial^2 F}{\partial r^2} + \\ +(r-\delta)H\frac{\partial F}{\partial H} + \kappa(\theta - r)\frac{\partial F}{\partial r} - rF = 0$$

N. J. Sharp, D. P. Newton and P. W. Duck, An Improved Fixed-Rate Mortgage Valuation Methodology with Interacting Prepayment and Default Options. Journal of Real Estate Finance and Economics, 19 (2008), 49-67

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Is geometric Brownian motion suitable for house prices evolution in certain market conditions?

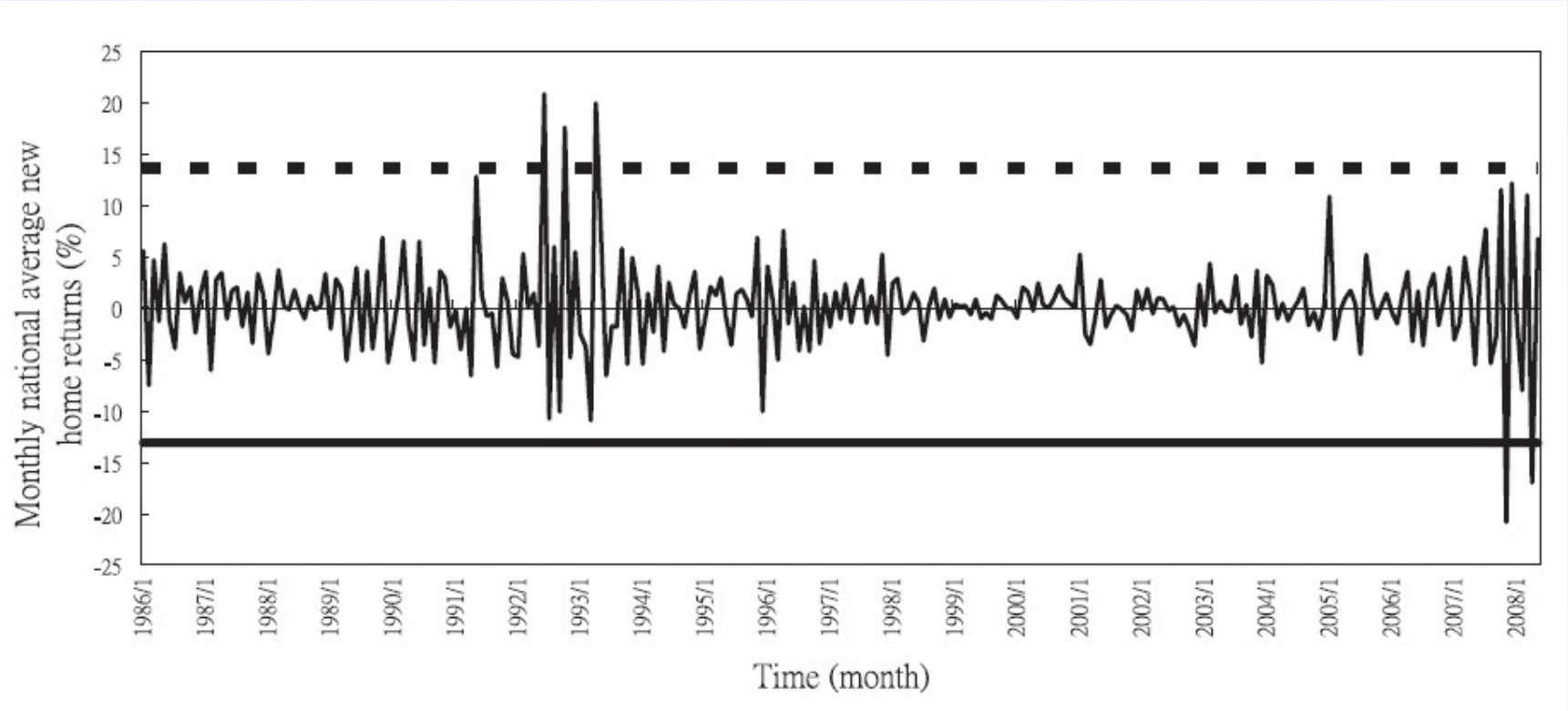

Figure: U.S. Time Series of new home price returns for single family

Motivation of jump-diffusion models (I)

- The assumption of geometric Brownian motion for house price results reasonable under relatively stable market dynamics
- However, for example time series of monthly house prices from January 1986 to June 2008 show several situations of jumps in house prices
- Subprime crisis caused significantly downward jumps since November 2007
- Other abnormal shocks are *"Black Wendesday"* in September 1992 or *"Irak disarmament crisis"* in July 1993, when U.S. Federal Reserve adapted an expansionary monetary policy

Motivation of jump-diffusion models (II)

M. C. Cheng, C. C. Chang, S. K. Lin and S. D. Yu, Estimation of housing price jump risks ad their impact on the valuation of mortgage insurance contracts. Journal of Risk and Insurance, 77 (2010), 399-422

- The authors estimate parameters of a general jump-diffusion model using expectation maximum (EM) gradient algorithms based on U.S. housing price data
- Empirical results show the likehood ratio test (LRT) rejects the model without jumps at significance level of 99 percent for new home prices, although not in the case of second hand houses
- Thus, actually the house price evolution contains two parts: continuous diffusion and discontinuous jumps

Stochastic model for the house value with jumps (I)

Evolution of the logarithmic house value, $Z_t = \ln(H_t)$ with jumps

$$dZ_t = \left(r_t - \frac{\sigma_H^2}{2} - \delta - \widetilde{\lambda}\widetilde{\kappa}\right) dt + \sigma_H dX_t^H + d\left(\sum_{i=1}^{N_t} V_i\right)$$

where

- δ : the 'dividend-type' per unit service flow provided by the house,
- σ_H : the house-price volatility,
- X_t^H : the standardised Wiener process for house price,
- $(N_t)_{t\geq 0}$: a Poisson process with parameter $\widetilde{\lambda}$
- (V_i) : a sequence of i.i.d random variables, with $\widetilde{\kappa} = E(\exp(V_i) - 1)$.

Stochastic model for the house value with jumps (and II)

Merton jump-diffusion model (1976)

- $(V_i) \sim N(\mu_j, \gamma_j^2)$ with the density

$$\nu_m(y) = \frac{1}{\gamma_j\sqrt{2\pi}} \exp\left(-\frac{(y-\mu_j)^2}{2\gamma_j^2}\right)$$

Kou jump-diffusion model (2002)

- $(V_i) \sim$ double-exponential distribution with the density

$$\nu_k(y) = \begin{cases} q\alpha_2 \exp(\alpha_2 y), & y < 0 \\ p\alpha_1 \exp(-\alpha_1 y), & y \geq 0 \end{cases}$$

with p, q, α_1 and α_2 positive constants with $p + q = 1$ and $\alpha_1 > 1$

PIDE for any asset depending on house price and interest rate (I)

Evolution of the house value H_t with jumps

$$dH_t = (r_t - \delta - \widetilde{\lambda}\widetilde{\kappa})H_t dt + \sigma_H dX_t^H + H_t d\left(\sum_{i=1}^{N_t}(Y_i - 1)\right)$$

where $Y_i = \exp(V_i)$.

PIDE in $(0, T) \times \Omega$

$$\frac{\partial F}{\partial t} + \frac{1}{2}\sigma_H^2 H^2 \frac{\partial^2 F}{\partial H^2} + \rho\sigma_H\sigma_r H\sqrt{r}\frac{\partial^2 F}{\partial H \partial r} + \frac{1}{2}\sigma_r^2 r \frac{\partial^2 F}{\partial r^2} + (r-\delta)H\frac{\partial F}{\partial H} + \kappa(\theta - r)\frac{\partial F}{\partial r} - rF$$
$$+ \int_{-\infty}^{\infty} \widetilde{\lambda}\left[F(t, H\exp(y), r) - F(t, H, r) - H(\exp(y) - 1)\frac{\partial F(t,H,r)}{\partial H}\right]\nu(y)dy = 0$$

$\nu(y) = \nu_m(y)$ for Merton model and $\nu(y) = \nu_k(y)$ for Kou one

PIDE for any asset depending on house price and interest rate (and II)

By considering that:

$$\int_{-\infty}^{\infty} \nu(y)dy = 1, \quad \int_{-\infty}^{\infty} \exp(y)\nu_m(y)dy = e^{\mu_j+\gamma_j^2/2}, \quad \int_{-\infty}^{\infty} \exp(y)\nu_k(y)dy = \frac{p\alpha_1}{\alpha_1-1} + \frac{q\alpha_2}{\alpha_2+1}$$

PIDE in $(0, T) \times \Omega$

$$\frac{\partial F}{\partial t} + \frac{1}{2}\sigma_H^2 H^2 \frac{\partial^2 F}{\partial H^2} + \rho\sigma_H\sigma_r H\sqrt{r}\frac{\partial^2 F}{\partial H \partial r} + \frac{1}{2}\sigma_r^2 r \frac{\partial^2 F}{\partial r^2} + (r - \delta - \widetilde{\lambda}\widetilde{\kappa})H\frac{\partial F}{\partial H} +$$
$$\kappa(\theta - r)\frac{\partial F}{\partial r} - (r + \widetilde{\lambda})F + \widetilde{\lambda}\int_{-\infty}^{\infty} F(t, H\exp(y), r)\nu(y)dy = 0$$

where $\widetilde{\kappa} = e^{\mu_j+\gamma_j^2/2} - 1$ (Merton) or $\widetilde{\kappa} = \frac{p\alpha_1}{\alpha_1-1} + \frac{q\alpha_2}{\alpha_2+1} - 1$ (Kou)

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Mortgage contract characteristics (I)

- Payments dates T_m, $m = 1, ..., M$
- Mortgage payment MP:

$$MP = \frac{(c/12)(1 + c/12)^M P(0)}{(1 + c/12)^M - 1}$$

- Outstanding balance P:

$$P(m-1) = \frac{((1 + c/12)^M - (1 + c/12)^{m-1})P(0)}{(1 + c/12)^M - 1}$$

where

- c is the fixed yearly contract rate,
- $P(0)$ is the initial amount loaned to the borrower
- M is the term of the loan in months

Mortgage contract characteristics (and II)

- $\Delta T_m = T_m - T_{m-1}$: total time in month m
- $t_m = t - T_{m-1}$: time elapsed at month m
- $\tau_m = \Delta T_m - t_m$: time until the payment date in month m

- Total debt payment TD in case of early prepayment:

$$TD(\tau_m) = (1 + \Psi)(1 + c(\Delta T_m - \tau_m))P(m - 1)$$

where Ψ is the prepayment penalty

IVP in the unbounded domain (I)

PDE in $(0, \Delta T_m) \times \Omega$

$$-\frac{\partial F}{\partial \tau_m} + \frac{1}{2}\sigma_H^2 H^2 \frac{\partial^2 F}{\partial H^2} + \rho\sigma_H\sigma_r H\sqrt{r}\frac{\partial^2 F}{\partial H \partial r} + \frac{1}{2}\sigma_r^2 r \frac{\partial^2 F}{\partial r^2} + (r-\delta)H\frac{\partial F}{\partial H} + \kappa(\theta - r)\frac{\partial F}{\partial r} - rF = 0$$

PIDE in $(0, \Delta T_m) \times \Omega$

$$-\frac{\partial F}{\partial \tau_m} + \frac{1}{2}\sigma_H^2 H^2 \frac{\partial^2 F}{\partial H^2} + \rho\sigma_H\sigma_r H\sqrt{r}\frac{\partial^2 F}{\partial H \partial r} + \frac{1}{2}\sigma_r^2 r \frac{\partial^2 F}{\partial r^2} + (r-\delta-\widetilde{\lambda}\widetilde{\kappa})H\frac{\partial F}{\partial H} + \kappa(\theta - r)\frac{\partial F}{\partial r} - (r+\widetilde{\lambda})F + \widetilde{\lambda}\int_{-\infty}^{\infty} F(t, H\exp(y), r)\nu(y)dy = 0$$

IVP in the unbounded domain (II)

- Contract value ($F(\tau_m, H, r) = V(\tau_m, H, r)$):
  - at maturity

$$V(\tau_M = 0, H, r) = \min(MP, H)$$

  - at other payment dates ($1 \leq m \leq M - 1$):

$$V(\tau_m = 0, H, r) = \min(V(\tau_{m+1} = T_{m+1}, H, r) + MP, H)$$

IVP in the unbounded domain (III)

- Insurance value ($F(\tau_m, H, r) = I(\tau_m, H, r)$):
  - at maturity

$$I(\tau_M = 0, H, r) = \begin{cases} \min(\gamma(MP - H), \Gamma) & \text{(Default)} \\ 0 & \text{(No default)} \end{cases}$$

  - at other payment dates ($1 \leq m \leq M - 1$):

$$I(\tau_m = 0, H, r) = \begin{cases} \min(\gamma[TD(\tau_m = 0) - H], \Gamma) & \text{(Default)} \\ I(\tau_{m+1} = \Delta T_{m+1}, H, r) & \text{(No default)} \end{cases}$$

where γ is a fraction of the total loss and Γ is the maximum indemnity

IVP in the unbounded domain (and IV)

- Coinsurance value ($F(\tau_m, H, r) = CI(\tau_m, H, r)$):
  - at maturity

$$CI(\tau_M = 0, H, r) = \begin{cases} \max((1-\gamma)(MP - H), (MP - H) - \Gamma) & \text{(Default)} \\ 0 & \text{(No default)} \end{cases}$$

  - at other payment dates ($1 \leq m \leq M - 1$):

$$CI(\tau_m = 0, H, r) = \begin{cases} \max((1-\gamma)[TD(\tau_m = 0) - H], [TD(\tau_m = 0) - H] - \Gamma) \\ \text{(Default)} \\ CI(\tau_{m+1} = \Delta T_{m+1}, H, r) & \text{(No default)} \end{cases}$$

Arbitrage condition

Equilibrium condition at origination:

$$V(\tau_1 = \Delta T_1, H_{initial}, r_{initial}; \Psi, c) + I(\tau_1 = \Delta T_1, H_{initial}, r_{initial}; \Psi, c) = (1-\xi)P(0)$$

where

- $\xi P(0)$ is the arrangement fee,
- Ψ is the prepayment penalty
- and the contract rate c is the only free parameter. Its value can be obtained by using a **variable secant method**.

J. B. Kau, D. C. Keenan, W. J. Muller III and J. F. Epperson. The Valuation at Origination of Fixed-Rate Mortgages with Default and Prepayment. Journal of Real Estate Finance and Economics, 11 (1995), 5-36

Free boundary problem (I)

Let us consider the following operators:

$$\begin{aligned}\mathcal{L}_1[V] &= -\frac{\partial V}{\partial \tau_m} + \frac{1}{2}\sigma_H^2 H^2 \frac{\partial^2 V}{\partial H^2} + \rho\sigma_H\sigma_r H\sqrt{r}\frac{\partial^2 V}{\partial H \partial r} + \frac{1}{2}\sigma_r^2 r \frac{\partial^2 V}{\partial r^2} + \\ &+ (r-\delta)H\frac{\partial V}{\partial H} + \kappa(\theta - r)\frac{\partial V}{\partial r} - rV\end{aligned}$$

$$\begin{aligned}\mathcal{L}_2[V] &= -\frac{\partial V}{\partial \tau_m} + \frac{1}{2}\sigma_H^2 H^2 \frac{\partial^2 V}{\partial H^2} + \rho\sigma_H\sigma_r H\sqrt{r}\frac{\partial^2 V}{\partial H \partial r} + \frac{1}{2}\sigma_r^2 r \frac{\partial^2 V}{\partial r^2} + \\ &(r-\delta-\widetilde{\lambda}\widetilde{\kappa})H\frac{\partial V}{\partial H} + \kappa(\theta - r)\frac{\partial V}{\partial r} - (r+\widetilde{\lambda})F \\ &+\widetilde{\lambda}\int_{-\infty}^{\infty} V(t, H\exp(y), r)\nu(y)dy\end{aligned}$$

Free boundary problem (and II)

So, the free boundary problem associated with the valuation of the mortgage contract, can be reduced to the linear complementarity problem:

$$\mathcal{L}[V] \leq 0, \quad (TD(\tau_m) - V(\tau_m, H, r)) \geq 0, \quad (\mathcal{L}[V])(TD(\tau_m) - V(\tau_m, H, r)) = 0$$

where $\mathcal{L}[V] = \mathcal{L}_1[V]$ or $\mathcal{L}[V] = \mathcal{L}_2[V]$

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Divergence form in the unbounded domain (I)

Let us consider $\rho = 0$

- In the absence of jumps

$$\mathcal{L}_1[F] = \partial_{\tau_m} F - Div(A\nabla F) + \vec{v} \cdot \nabla F + lF$$

$$A(H,r) = \begin{pmatrix} \frac{1}{2}\sigma_H^2 H^2 & 0 \\ 0 & \frac{1}{2}\sigma_r^2 r \end{pmatrix}$$

$$\vec{v}(H,r) = \begin{pmatrix} (\sigma_H^2 - r + \delta)H \\ (\frac{1}{2}\sigma_r^2 - \kappa(\theta - r)) \end{pmatrix}$$

$$l(\tau_m, H, r) = r$$

Divergence form in the unbounded domain (II)

- In the presence of jumps

$$\mathcal{L}_2[F] = \partial_{\tau_m} F - Div(A_j \nabla F) + \vec{v}_j \cdot \nabla F + l_j F - \widetilde{\lambda} \int_{-\infty}^{\infty} F(\tau_m, H \exp(y), r) \nu(y) dy$$

$$A_j(H, r) = \begin{pmatrix} \frac{1}{2}\sigma_H^2 H^2 & 0 \\ 0 & \frac{1}{2}\sigma_r^2 r \end{pmatrix}$$

$$\vec{v}_j(H, r) = \begin{pmatrix} (\sigma_H^2 - r + \delta + \widetilde{\lambda}\widetilde{\kappa})H \\ (\frac{1}{2}\sigma_r^2 - \kappa(\theta - r)) \end{pmatrix}$$

$$l_j(\tau_m, H, r) = r + \widetilde{\lambda}$$

Divergence form in the unbounded domain (and III)

Insurance and coinsurance ($F = I, CI$): Cauchy problem

$$\begin{cases} \mathcal{L}[F] = f, \text{in } (0, \Delta T_m) \times \Omega, \\ +\text{appropriate initial conditions for each month} \end{cases}$$

Mortgage contract value ($F = V$): complementarity problem

$$\begin{cases} \max\{\mathcal{L}[F] - f, F - TD\} = 0, \text{in } (0, \Delta T_m) \times \Omega, \\ +\text{appropriate initial conditions for each month} \end{cases}$$

where

$$\mathcal{L}[F] = \mathcal{L}_1[F] \quad or \quad \mathcal{L}[F] = \mathcal{L}_2[F] \quad and \quad f(\tau_m, H, r) = 0$$

Main difficulties in the numerical solution

- Unbounded domain in house price and interest rate directions
  Localization+boundary conditions
- Unbounded domain of integration in the integral term
  Localization
- Diffusion matrix is degenerated
  (convection dominated problem)
  Higher order Lagrange/Galerkin methods
- Nonlinearities in the free boundary problem
  ALAS algorithm

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- Diffusion matrix is degenerated
  (convection dominated problem)
  Higher order Lagrange/Galerkin methods
- Nonlinearities in the free boundary problem
  ALAS algorithm

Some references for numerical techniques

- Higher order characteristics
  A. Bermúdez, M.R. Nogueiras, and C. Vázquez, *Numerical analysis of convection-diffusion-reaction problems with higher order characteristics/finite elements. Part I: Time discretization*, *SIAM J. Numer. Anal.*, 44 (2006) 1829–1853.
- Spatial discretization:
  A. Bermúdez, M. R. Nogueiras, and C. Vázquez, *Numerical analysis of convection-diffusion-reaction problems with higher order characteristics/finite elements. Part II: Fully discretized scheme and quadrature formulas*, *SIAM J. Numer. Anal.*, 44 (2006) 1854–1876.
- ALAS algorithm:
  T. Kärkkäinen, K. Kunisch, and P. Tarvainen, *Augmented Lagrangian active set methods for obstacle problems*, *J. Optim. Theory Appl.*, 119 (2003), pp. 499–533.

Formulation in the bounded domain without jumps (I)

We make the change of spatial variables: $x_1 = \frac{H}{H_\infty}$ and $x_2 = \frac{r}{r_\infty}$.
So, $\Omega^* = (0,1) \times (0,1)$

$\Gamma_i^- = \{(x_1, x_2) \in \partial\Omega^* \mid x_i = 0\}$, $\Gamma_i^+ = \{(x_1, x_2) \in \partial\Omega^* \mid x_i = 1\}$, $i = 1, 2$

- **Insurance and coinsurance ($J = I, CI$)**
  Find $J : [0, \Delta T_m] \times \Omega^* \to \mathbb{R}$ satisfying the PDE

$$\frac{\partial J}{\partial \tau_m} - Div(A\nabla J) + \vec{v} \cdot \nabla J + IJ = f \qquad \text{in } (0, \Delta T_m) \times \Omega^*$$

- **Mortgage contract value (V)**
  Find $V : [0, \Delta T_m] \times \Omega^* \to \mathbb{R}$ satisfying the PDE

$$\frac{\partial V}{\partial \tau_m} - Div(A\nabla V) + \vec{v} \cdot \nabla V + IV + P = f \qquad \text{in } (0, \Delta T_m) \times \Omega^*$$

  jointly with the complementarity conditions

$$V \leq TD, \quad P \geq 0, \quad P(TD - V) = 0 \ \text{ in } (0, \Delta T_m) \times \Omega^*$$

Formulation in the bounded domain without jumps (and II)

where

$$A(x_1, x_2) = \begin{pmatrix} \frac{1}{2}\sigma_H^2 x_1^2 & 0 \\ 0 & \frac{1}{2}\sigma_r^2 \frac{x_2}{r_\infty} \end{pmatrix}$$

$$\vec{v}(x_1, x_2) = \begin{pmatrix} (\sigma_H^2 - x_2 r_\infty + \delta)x_1 \\ (\frac{1}{2}\sigma_r^2 - \kappa(\theta - x_2 r_\infty))/r_\infty \end{pmatrix}$$

$$l(\tau_m, x_1, x_2) = x_2 r_\infty$$

$$f(\tau_m, x_1, x_2) = 0$$

Integral term localization

Change of variable: $\bar{x}_1 = log(x_1)$

$$\int_{-\infty}^{\infty} F(\tau_m, x_1 \exp(y), x_2)\nu(y)dy \approx \int_{y_{min}}^{y_{max}} \bar{F}(\tau_m, \bar{x}_1 + y, x_2)\nu(y)dy$$

where $\bar{F}(\tau_m, \bar{x}_1 + y, x_2) = F(\tau_m, \exp(\bar{x}_1 + y), x_2)$.
If we consider the discrete grid $0 = x_{1_0}, x_{1_1}, \cdots, x_{1_q} = 1$ then $y_{min} = log(x_{1_1})$ and $y_{max} = log(x_{1_q})$.

Y. D'Halluin, P. A. Forsyth and K. R. Vetzal, *Robust numerical methods for contingent claims under jump diffusion processes*, IMA J. Numer. Anal, 25 (2005), 87-112.

Formulation in the bounded domain with jumps (I)

- **Insurance and coinsurance ($J = I, CI$)**
  Find $J : [0, \Delta T_m] \times \Omega^* \to \mathbb{R}$ such that

$$\frac{\partial J}{\partial \tau_m} - Div(A_j \nabla J) + \vec{v}_j \cdot \nabla J + l_j J - \widetilde{\lambda} \int_{y_{min}}^{y_{max}} \bar{J}(\tau_m, \bar{x}_1 + y, x_2)\nu(y)dy = f$$

$$\text{in } (0, \Delta T_m) \times \Omega^*$$

- **Mortgage contract value (V)**
  Find $V : [0, \Delta T_m] \times \Omega^* \to \mathbb{R}$ satisfying the partial differential equation

$$\frac{\partial V}{\partial \tau_m} - Div(A_j \nabla V) + \vec{v}_j \cdot \nabla V + l_j V - \widetilde{\lambda} \int_{y_{min}}^{y_{max}} \bar{V}(\tau_m, \bar{x}_1 + y, x_2)\nu(y)dy + P = f$$

$$\text{in } (0, \Delta T_m) \times \Omega^*$$

the complementarity conditions

$$V \leq TD, \quad P \geq 0, \quad P(TD - V) = 0 \ \text{ in } (0, \Delta T_m) \times \Omega^*$$

Formulation in the bounded domain with jumps (and II)

where

$$A_j(x_1, x_2) = \begin{pmatrix} \frac{1}{2}\sigma_H^2 x_1^2 & 0 \\ 0 & \frac{1}{2}\sigma_r^2 \frac{x_2}{r_\infty} \end{pmatrix}$$

$$\vec{v}_j(x_1, x_2) = \begin{pmatrix} (\sigma_H^2 - x_2 r_\infty + \delta + \widetilde{\lambda}\widetilde{\kappa})x_1 \\ (\frac{1}{2}\sigma_r^2 - \kappa(\theta - x_2 r_\infty))/r_\infty \end{pmatrix}$$

$$l_j(\tau_m, x_1, x_2) = x_2 r_\infty + \widetilde{\lambda}$$

$$f(\tau_m, x_1, x_2) = 0$$

Boundary conditions

Moreover, we consider in both cases the previous initial conditions and the following boundary conditions

$$\frac{\partial J}{\partial x_1} = 0, \quad \frac{\partial V}{\partial x_1} = 0 \quad \text{on } (0, \Delta T_m) \times \Gamma_1^+$$
$$\frac{\partial J}{\partial x_1} = 0, \quad \frac{\partial V}{\partial x_2} = 0 \quad \text{on } (0, \Delta T_m) \times \Gamma_2^+$$

O.A. Oleinik, E. V. Radkevic, *Second order equations with nonnegative characteristic form*, A.M.S. And Plenum Press, 1973

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Fixed parameters in the mortgage valuation model

Economic framework	
Steady state spot rate, θ	10 %
Speed of reversion, κ	25 %
House service flow, δ	7.5%
Correlation coefficient, ρ	0
Contract specifications	
Initial value of the house, $H_{initial}$	100000€
Ratio of the loan to value	95 %
Initial estimate for contract rate, c_0	10%
Prepayment penalty, Ψ	5%
Insurance	
Guaranteed fraction of total loss, γ	80%
Cap, Γ	$20\% H_{initial}$

Table: Fixed parameters in the mortgage valuation model

Parameters of numerical methods

Computational domain	
H_∞	200000€
r_∞	40 %
Finite elements mesh data	
Number of elements	576
Number of nodes	2401
Time discretization	
Time steps per month	30
ALAS algorithm	
Parameter β	10000

Table: Parameters of numerical methods

Contract rate, mortgage, insurance and coinsurance value for $\sigma_r = 5\%$, $\sigma_H = 5\%$ without jumps in the house value

Loan (years)	spot rate r(0)	ξ	Contract rate c	Contract value V	Insurance I	Coinsurance CI
15	8%	0%	9.0839%	94549	449	112
		0.5%	8.9911%	94116	410	103
		1%	8.8992%	93663	386	96
		1.5%	8.8119%	93230	345	86
	10%	0%	10.0782%	94656	343	84
		0.5%	9.9696%	94208	317	79
		1%	9.8634%	93764	288	72
		1.5%	9.7579%	93316	260	66
	12%	0%	11.1662%	94691	309	76
		0.5%	11.0389%	94274	249	62
		1%	10.9203%	93870	181	45
		1.5%	10.8006%	93422	154	38
25	8%	0%	9.2605%	93961	1039	260
		0.5%	9.1876%	93549	974	243
		1%	9.1158%	93117	933	233
		1.5%	9.0453%	92677	899	225
	10%	0%	10.1258%	94314	685	171
		0.5%	10.0369%	93878	646	162
		1%	9.9440%	93417	632	158
		1.5%	9.8551%	92970	604	151
	12%	0%	11.1585%	94536	464	116
		0.5%	11.0462%	94126	399	101
		1%	10.9219%	93667	382	94

Contract rate, mortgage, insurance and coinsurance value for $\sigma_r = 10\%$, $\sigma_H = 5\%$ without jumps in the house value

Loan (years)	spot rate r(0)	ξ	Contract rate c	Contract value V	Insurance I	Coinsurance CI
15	8%	0%	9.3028%	94390	609	152
		0.5%	9.1741%	93959	566	141
		1%	9.0484%	93523	526	131
		1.5%	8.9184%	93064	511	128
	10%	0%	10.5172%	94506	494	124
		0.5%	10.3544%	94065	459	115
		1%	10.1925%	93621	429	107
		1.5%	10.0424%	93196	378	95
	12%	0%	11.8193%	94610	389	97
		0.5%	11.6207%	94161	364	91
		1%	11.4324%	93723	327	81
		1.5%	11.2617%	93270	305	76
25	8%	0%	9.5142%	93778	1222	306
		0.5%	9.3969%	93315	1209	302
		1%	9.2833%	92847	1202	300
		1.5%	9.1746%	92387	1187	296
	10%	0%	10.6232%	94102	898	224
		0.5%	10.4877%	93688	836	209
		1%	10.3441%	93235	815	203
		1.5%	10.2052%	92780	795	198
	12%	0%	11.8641%	94344	655	163
		0.5%	11.6778%	93885	639	159
		1%	11.4993%	93430	620	155

Contract rate, mortgage, insurance and coinsurance value for $\sigma_r = 5\%$, $\sigma_H = 10\%$ without jumps in the house value

Loan (years)	spot rate r(0)	ξ	Contract rate c	Contract value V	Insurance I	Coinsurance CI
15	8%	0%	9.0078%	92650	2350	587
		0.5%	8.9084%	92242	2282	571
		1%	8.8132%	91845	2205	551
		1.5%	8.7195%	91446	2129	532
	10%	0%	10.0154%	92984	2015	503
		0.5%	9.8983%	92565	1960	490
		1%	9.7861%	92154	1896	474
		1.5%	9.6801%	91748	1826	456
	12%	0%	11.1181%	93270	1730	432
		0.5%	10.9775%	92849	1676	418
		1%	10.8459%	92427	1622	405
		1.5%	10.7241%	92015	1559	389
25	8%	0%	9.2191%	91407	3594	898
		0.5%	9.1386%	90991	3533	882
		1%	9.0585%	90565	3484	870
		1.5%	8.9818%	90144	3430	857
	10%	0%	10.0815%	91997	3003	751
		0.5%	9.9881%	91590	2934	733
		1%	9.9022%	91204	2845	711
		1.5%	9.8104%	90778	2797	699
	12%	0%	11.1048%	92532	2468	624
		0.5%	10.9742%	92090	2434	608
		1%	10.8564%	91675	2376	592

Mortgage value at origination without jumps in the house value

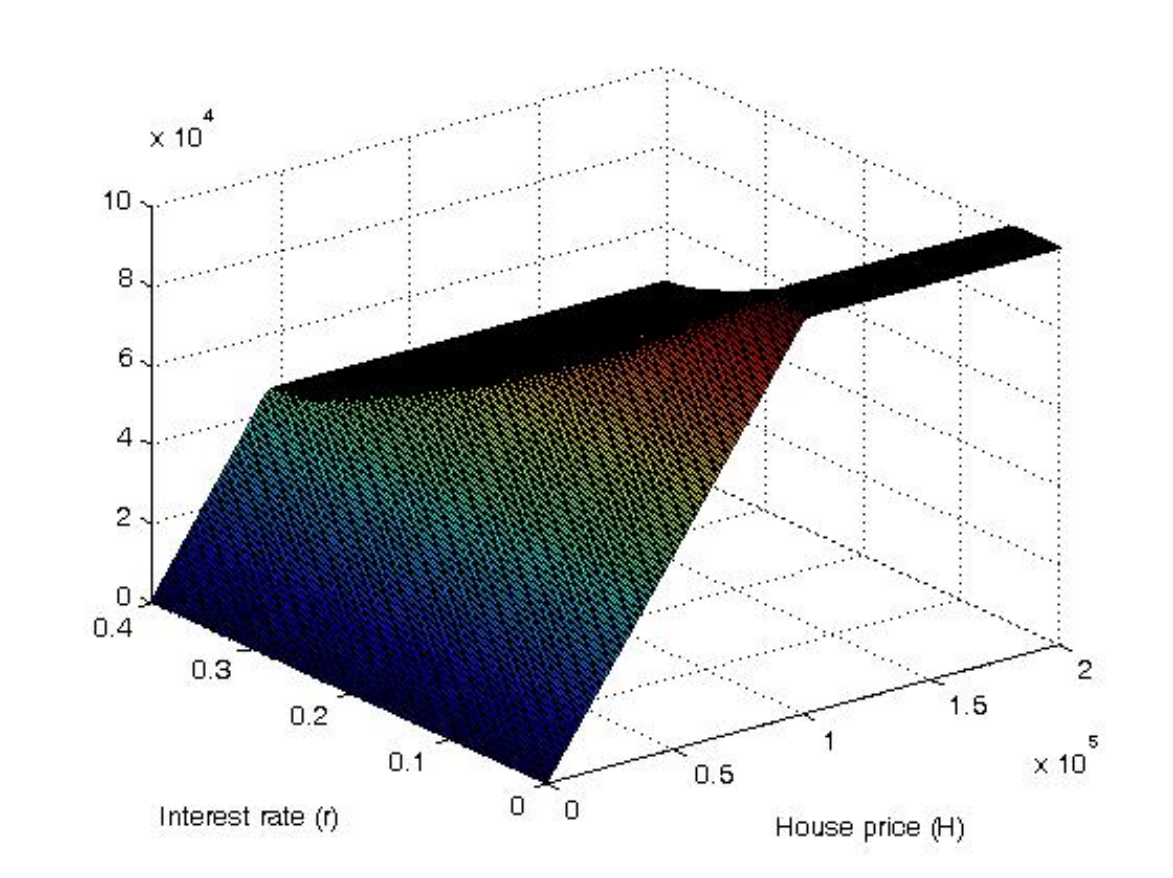

Figure: Mortgage value at origination when the arrangement fee is 0.5%, the early exercise penalty takes the value of 5%, the contract rate is 9.3969%, the interest rate volatility is 10%, the house price volatility is 5%, the maturity of the contract is 25 years and the spot rate is 8%

Insurance value at origination without jumps in the house value

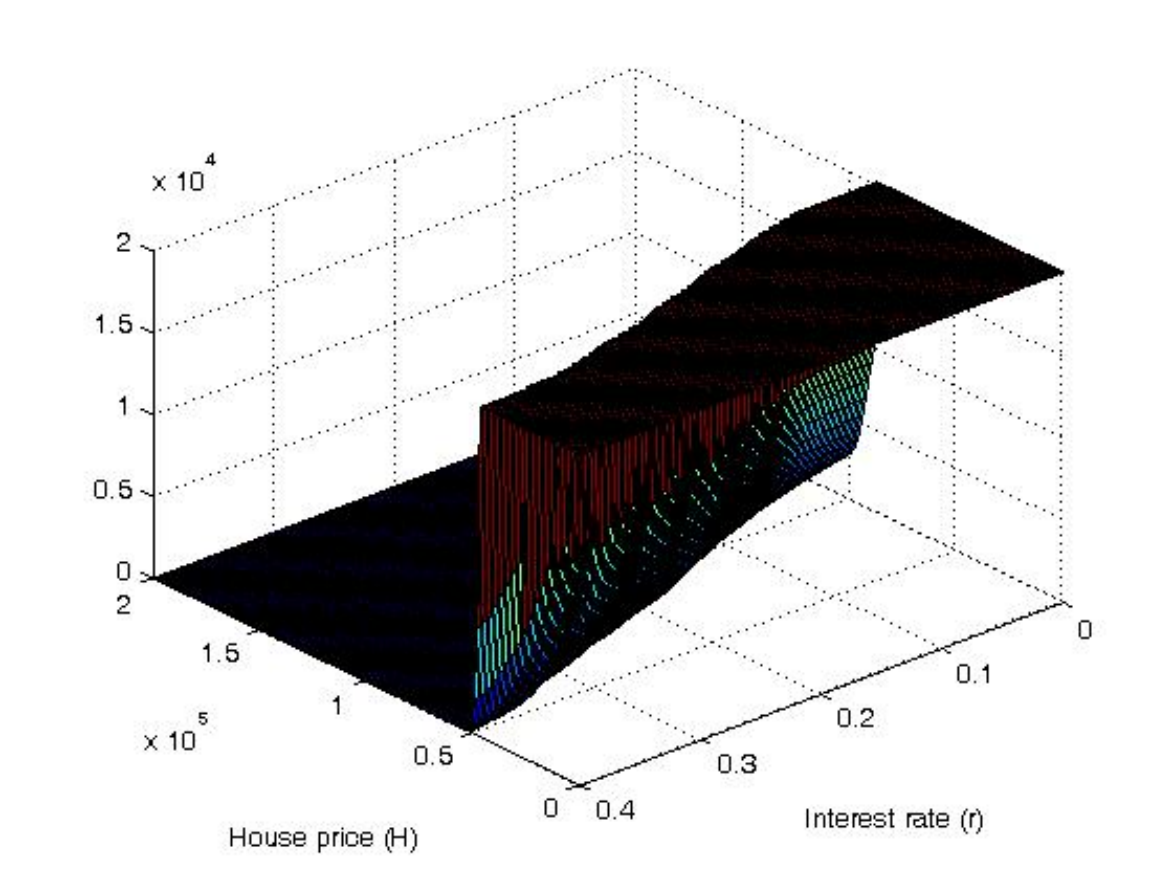

Figure: Insurance value at origination when the arrangement fee is 0.5%, the early exercise penalty takes the value of 5%, the contract rate is 9.3969%, the interest rate volatility is 10%, the house price volatility is 5%, the maturity of the contract is 25 years and the spot rate is 8%

Coinsurance value at origination without jumps in the house value

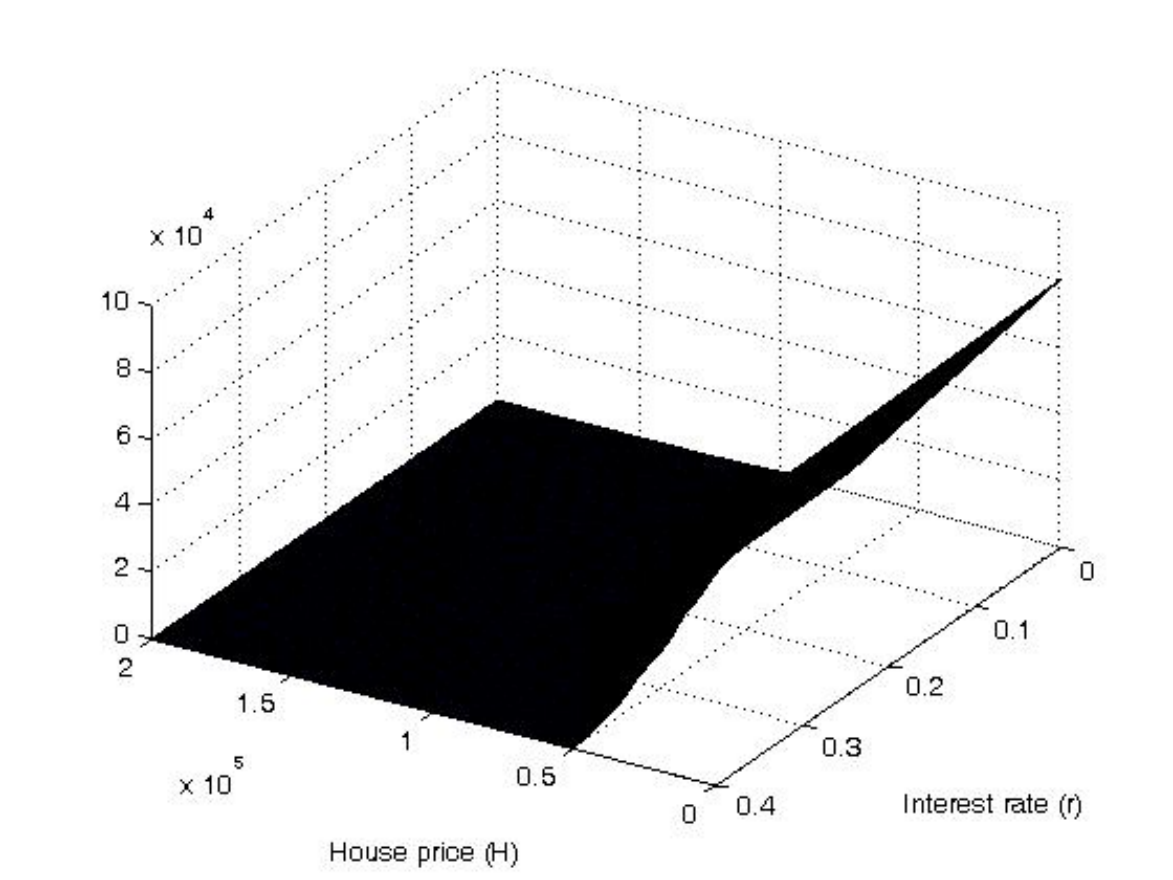

Figure: Coinsurance value at origination when the arrangement fee is 0.5%, the early exercise penalty takes the value of 5%, the contract rate is 9.3969%, the interest rate volatility is 10%, the house price volatility is 5%, the maturity of the contract is 25 years and the spot rate is 8%

Optimal prepayment boundary at origination without jumps in the house value

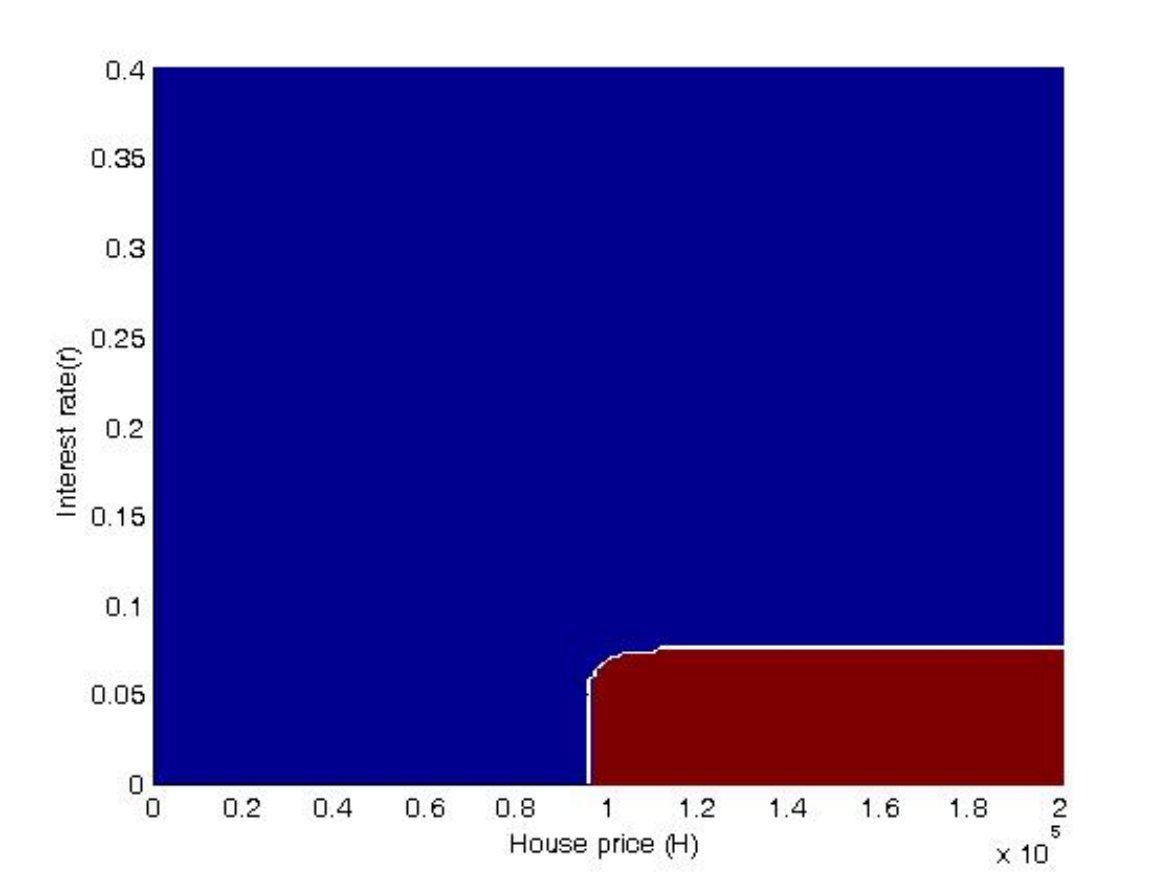

Figure: Optimal prepayment boundary at origination when the arrangement fee is 0.5%, the early exercise penalty takes the value of 5%, the contract rate is 9.3969%, the interest rate volatility is 10%, the house price volatility is 5%, the maturity of the contract is 25 years and the spot rate is 8%

Parameters in the jump-diffusion models

Parameters in the jump-diffusion models	
Parameter of Poisson process, $\widetilde{\lambda}$	0.1
Mean of jump size (Merton), μ_j	-0.1
Standard deviations of jump size (Merton), γ_j	0.45
Probability of upward jump (Kou), p	0.3445
Parameter (Kou), α_1	3.0465
Parameter (Kou), α_2	3.0775

Table: Parameters in the jump-diffusion models

Contract rate, mortgage, insurance and coinsurance value for $\sigma_r = 5\%$, $\sigma_H = 5\%$ under Merton jump-diffusion model for the house value

Loan (years)	spot rate r(0)	ξ	Contract rate c	Contract value V	Insurance I	Coinsurance CI
15	8%	0%	14.4301%	91730	3270	2402
		0.5%	14.3121%	91282	3243	2402
		1%	14.1966%	90840	3210	2402
		1.5%	14.0815%	90396	3179	2402
	10%	0%	15.4554%	92050	2950	2190
		0.5%	15.3245%	91588	2937	2190
		1%	15.1965%	91132	2918	2190
		1.5%	15.0698%	90674	2901	2188
	12%	0%	16.5677%	92360	2640	1992
		0.5%	16.4168%	91892	2633	1998
		1%	16.2706%	91428	2622	1998
		1.5%	16.1271%	90960	2615	1998

Contract rate, mortgage, insurance and coinsurance value for $\sigma_r = 5\%$, $\sigma_H = 5\%$ under Kou jump-diffusion model for the house value

Loan (years)	spot rate r(0)	ξ	Contract rate c	Contract value V	Insurance I	Coinsurance CI
15	8%	0%	14.2355%	92090	2910	2092
		0.5%	14.1191%	91647	2878	2090
		1%	14.0045%	91202	2848	2090
		1.5%	13.8920%	90759	2816	2090
	10%	0%	15.2618%	92404	2596	1909
		0.5%	15.1339%	91949	2576	1910
		1%	15.0078%	91492	2558	1910
		1.5%	14.8838%	91036	2539	1908
	12%	0%	16.3824%	92714	2286	1744
		0.5%	16.2317%	92244	2281	1744
		1%	16.0861%	91778	2272	1748
		1.5%	15.9438%	91314	2261	1746

Optimal prepayment boundary at origination under Merton jump-diffusion model

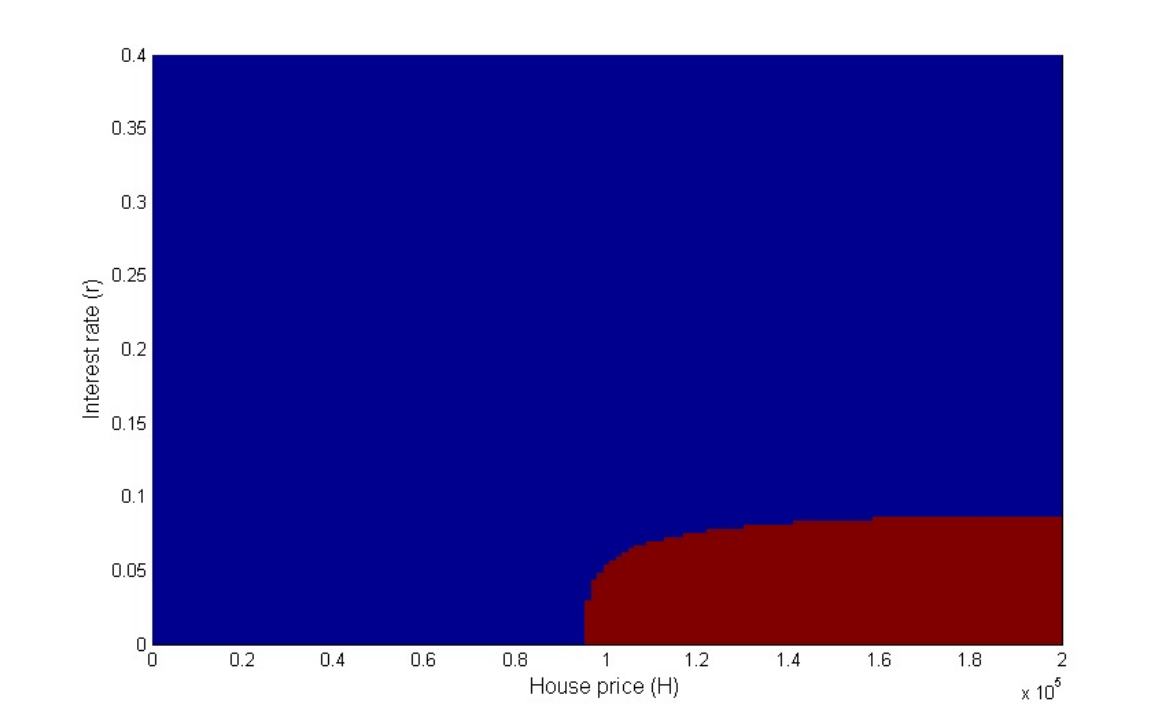

Figure: Optimal prepayment boundary at origination when the arrangement fee is 0.5%, the early exercise penalty takes the value of 5%, the contract rate is 14.5892%, the interest rate volatility is 10%, the house price volatility is 5%, the maturity of the contract is 25 years and the spot rate is 8%

Optimal prepayment boundary at origination under Kou jump-diffusion model

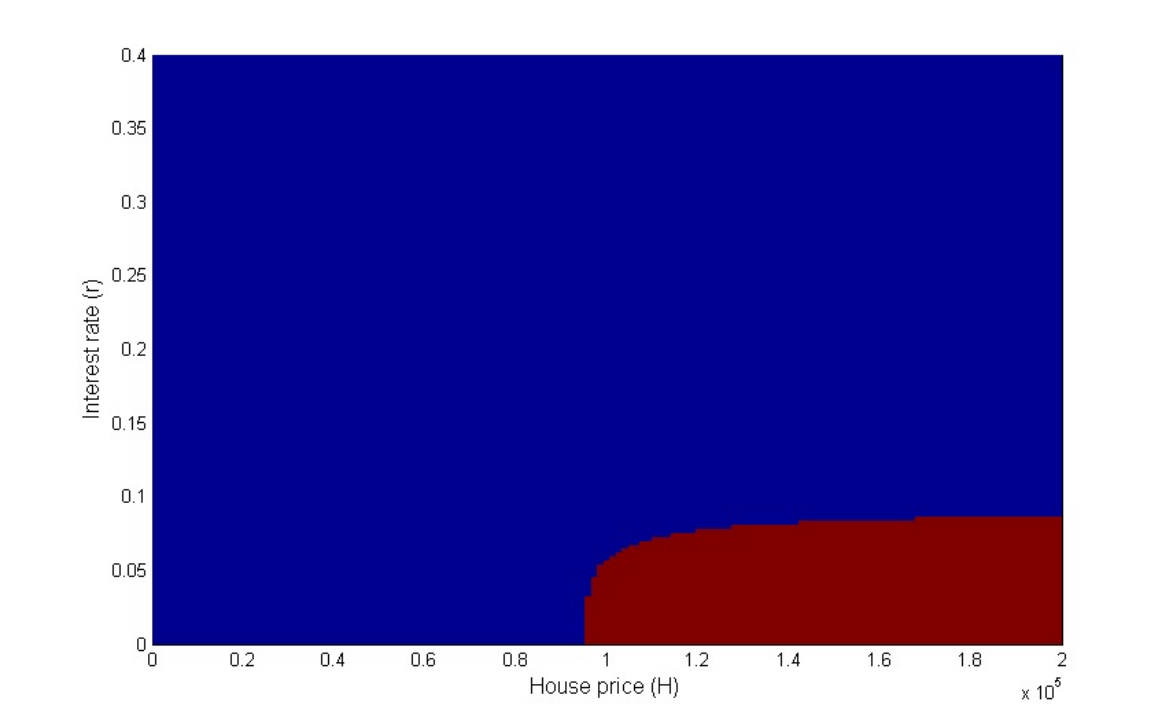

Figure: Optimal prepayment boundary at origination when the arrangement fee is 0.5%, the early exercise penalty takes the value of 5%, the contract rate is 14.3926%, the interest rate volatility is 10%, the house price volatility is 5%, the maturity of the contract is 25 years and the spot rate is 8%

Outline

1. Fixed rate mortgage contract. Objectives
2. Mathematical models
   - PDE model
   - PIDE model
   - Additional mortgage characteristics
3. Numerical methods for PDE and PIDE models
4. Numerical results
5. Conclusions

Conclusions (I)

- A PDE model to obtain the value of a fixed-rate mortgage with the prepayment and default options without jumps in the house value is proposed.
- A PIDE model to obtain the value of a fixed-rate mortgage with the prepayment and default with jumps in the house value is proposed.
- We consider Merton and Kou jump-diffusion processes.
- The stochastic variables are the house price and the interest rate.
- Insurance and coinsurance: initial boundary value problem for each month.

Conclusions (and II)

- Prepayment option in the mortgage contract: a complementarity problem for each month.
- The equilibrium interest rate is adjusted by using an iterative method.
- The mathematical models are solved by using appropriate numerical methods and several numerical results are presented.

Further information

M. C. Calvo-Garrido and C. Vázquez, A new numerical method for pricing Fixed-Rate Mortgages with prepayment and default options, *International Journal of Computer Mathematics*, 93 (2016), 5, 761-780

M.C. Calvo-Garrido and C. Vázquez, Effects of jump-diffusion models for the house price dynamics in the pricing of fixed-rate mortgages, insurance and coinsurance, *Applied Mathematics and Computation*, 271 (2015), 730-742.

Thank you for your attention!

Dynamic hedging methodology (I)

We build a portfolio

$$\Pi = F_1 - \Delta_2 F_2 - \Delta_1 H$$

Initially, the variation of the portfolio value in $[t, t + dt]$ is given by

$$d\Pi = dF_1 - \Delta_2 dF_2 - \Delta_1 dH$$

However, due to the dividend yield δ, the portfolio must change by an amount

$$-\delta H \Delta_1 dt$$

Thus, actually the change of the portfolio value during $[t, t + dt]$ is

$$d\Pi = dF_1 - \Delta_2 dF_2 - \Delta_1 (dH + \delta H dt)$$

Dynamic hedging methodology (II)

Thus, Π turns out to be risk free if we choose:

$$\Delta_2 = \frac{\partial F_1/\partial r}{\partial F_2/\partial r}$$

$$\Delta_1 = \frac{\partial F_1}{\partial H} - \Delta_2 \frac{\partial F_2}{\partial H}$$

Dynamic hedging methodology (III)

- For the previous choice of Δ_1 and Δ_2, we have:

$$
\begin{aligned}
d\Pi \quad = \quad & \left[\frac{\partial F_1}{\partial t} + \frac{1}{2}\left(\sigma_H^2 H^2 \frac{\partial^2 F_1}{\partial H^2} + 2\rho\sigma_H\sigma_r H\sqrt{r}\frac{\partial^2 F_1}{\partial H \partial r} + \sigma_r^2 r \frac{\partial^2 F_1}{\partial r^2} \right) - \delta H \frac{\partial F_1}{dH} \right. \\
& \left. - \frac{\partial F_1/\partial r}{\partial F_2/\partial r}\left(\frac{\partial F_2}{\partial t} + \frac{1}{2}\left(\sigma_H^2 H^2 \frac{\partial^2 F_2}{\partial H^2} + 2\rho\sigma_H\sigma_r H\sqrt{r}\frac{\partial^2 F_2}{\partial H \partial r} + \sigma_r^2 r \frac{\partial^2 F_2}{\partial r^2} \right) - \delta H \frac{\partial F_2}{dH} \right) \right] dt
\end{aligned}
$$

- By using no-arbitrage assumption, we have:

$$d\Pi = r\Pi dt$$

- Identifying both quantities we get...

Dynamic hedging methodology (IV)

$$\frac{1}{\partial F_1/\partial r}\left(\frac{\partial F_1}{\partial t}+\frac{1}{2}\sigma_H^2 H^2\frac{\partial^2 F_1}{\partial H^2}+\rho\sigma_H\sigma_r H\sqrt{r}\frac{\partial^2 F_1}{\partial H\partial r}+\frac{1}{2}\sigma_r^2 r\frac{\partial^2 F_1}{\partial r^2}+(r-\delta)H\frac{\partial F_1}{\partial H}-rF_1\right) =$$

$$\frac{1}{\partial F_2/\partial r}\left(\frac{\partial F_2}{\partial t}+\frac{1}{2}\sigma_H^2 H^2\frac{\partial^2 F_2}{\partial H^2}+\rho\sigma_H\sigma_r H\sqrt{r}\frac{\partial^2 F_2}{\partial H\partial r}+\frac{1}{2}\sigma_r^2 r\frac{\partial^2 F_2}{\partial r^2}+(r-\delta)H\frac{\partial F_2}{\partial H}-rF_2\right)$$

Dynamic hedging methodology (and V)

Both sides are independent of maturity. So,

$$\frac{1}{\partial F/\partial r}\left(\frac{\partial F}{\partial t}+\frac{1}{2}\sigma_H^2 H^2\frac{\partial^2 F}{\partial H^2}+\rho\sigma_H\sigma_r H\sqrt{r}\frac{\partial^2 F}{\partial H\partial r}+\right.$$
$$\left.+\frac{1}{2}\sigma_r^2 r\frac{\partial^2 F}{\partial r^2}+(r-\delta)H\frac{\partial F}{\partial H}-rF\right) \quad = \quad a(t,H,r)$$

where $a(t,H,r)=-\kappa(\theta-r)$.

J. A. Azevedo-Pereira, D. P. Newton and D. A. Paxson, UK Fixed Rate Repayment Mortgage and Mortgage Indemnity Valuation, Real Estate Economics, 30 (2002), 185-211

Time discretization: method of characteristics (I)

Vector associated with the first order derivative terms (velocity field):

- In the absence of jumps:

$$\vec{v}_1 = \vec{v} = \begin{pmatrix} (\sigma_H^2 - x_2 r_\infty + \delta) x_1 \\ (\frac{1}{2}\sigma_r^2 - \kappa(\theta - x_2 r_\infty))/r_\infty \end{pmatrix}$$

- In the presence of jumps:

$$\vec{v}_1 = \vec{v}_j = \begin{pmatrix} (\sigma_H^2 - x_2 r_\infty + \delta + \widetilde{\lambda}\widetilde{\kappa}) x_1 \\ (\frac{1}{2}\sigma_r^2 - \kappa(\theta - x_2 r_\infty))/r_\infty \end{pmatrix}$$

Characteristic curves through $\mathbf{x} = (x_1, x_2)$ at time τ_m^{n+1}: $X(\mathbf{x}, \tau_m^{n+1}; s)$

$$\frac{\partial}{\partial s} X(\mathbf{x}, \tau_m^{n+1}; s) = \vec{v}_1(X(\mathbf{x}, \tau_m^{n+1}; s)), \; X(\mathbf{x}, \tau_m^{n+1}; \tau_m^{n+1}) = \mathbf{x}$$

Time discretization: method of characteristics (II)

Characteristic curves can be exactly computed:

- In the absence of jumps

$$
\begin{aligned}
X_1^n(\mathbf{x}) &= x_1 exp(-(\sigma_H^2 + \delta + \frac{\sigma_r^2}{2\kappa} - \theta)\Delta\tau_m) \times \\
& \quad exp((\frac{-x_2 r_\infty}{\kappa} - \frac{\sigma_r^2}{2\kappa^2} + \frac{\theta}{\kappa})(exp(-\kappa\Delta\tau_m) - 1))
\end{aligned}
$$

$$
X_2^n(\mathbf{x}) = (-\frac{\sigma_r^2}{2\kappa r_\infty} + \frac{\theta}{r_\infty})(1 - exp(-\kappa\Delta\tau_m)) + x_2 exp(-\kappa\Delta\tau_m)
$$

where $X^n(\mathbf{x}) := X(\mathbf{x}, \tau_m^{n+1}; \tau_m^n)$

M. C. Calvo-Garrido and C. Vázquez, *A new numerical method for pricing Fixed-Rate Mortgages with prepayment and default options*, International Journal of Computer Mathematics, 93 (2016), 5, 761-780

Time discretization: method of characteristics (and III)

- In the presence of jumps

$$
\begin{aligned}
X_1^n(\mathbf{x}) &= x_1 exp(-(\sigma_H^2 + \delta + \frac{\sigma_r^2}{2\kappa} - \theta + \widetilde{\lambda}\widetilde{\kappa})\Delta\tau_m) \times \\
& \quad exp((\frac{-x_2 r_\infty}{\kappa} - \frac{\sigma_r^2}{2\kappa^2} + \frac{\theta}{\kappa})(exp(-\kappa\Delta\tau_m) - 1))
\end{aligned}
$$

$$
X_2^n(\mathbf{x}) = (-\frac{\sigma_r^2}{2\kappa r_\infty} + \frac{\theta}{r_\infty})(1 - exp(-\kappa\Delta\tau_m)) + x_2 exp(-\kappa\Delta\tau_m)
$$

where $X^n(\mathbf{x}) := X(\mathbf{x}, \tau_m^{n+1}; \tau_m^n)$

M.C. Calvo-Garrido and C. Vázquez, Effects of jump-diffusion models for the house price dynamics in the pricing of fixed-rate mortgages, insurance and coinsurance, Applied Mathematics and Computation, 271 (2015), 730-742.

Crank-Nicolson-Characteristic (CN-char) scheme

Time step $\Delta\tau_m = \frac{\Delta T_m}{N}$
Time meshpoints $\tau_m^n = n\Delta\tau_m,\ n = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots, N.$

Characteristics for material derivative approximation:

$$\frac{DF}{D\tau_m} = \frac{F^{n+1} - F^n \circ X^n}{\Delta\tau_m}$$

where $X^n(\boldsymbol{x}) := X(\boldsymbol{x}, \tau_m^{n+1}; \tau_m^n)$

Crank-Nicolson around $\left(X(\boldsymbol{x}, \tau_m^{n+1}; \tau_m), \tau_m\right)$ for $\tau = \tau_m^{n+\frac{1}{2}}$

CN-char scheme for time discretization without jumps

For n=0,...,N-1, find F^{n+1} such that:

$$\frac{F^{n+1}(\mathbf{x}) - F^n(X^n(\mathbf{x}))}{\Delta\tau_m} - \frac{1}{2}Div(A\nabla F^{n+1})(\mathbf{x}) - \frac{1}{2}Div(A\nabla F^n)(X^n(\mathbf{x}))$$
$$+\frac{1}{2}(I\,F^{n+1})(\mathbf{x}) + \frac{1}{2}(I\,F^n)(X^n(\mathbf{x})) = 0$$

where $F = I, CI, V$

CN-char scheme for time discretization with jumps

For n=0,...,N-1, find F^{n+1} such that:

$$\frac{F^{n+1}(\mathbf{x}) - F^n(X^n(\mathbf{x}))}{\Delta\tau_m} - \frac{1}{2}Div(A_j\nabla F^{n+1})(\mathbf{x}) - \frac{1}{2}Div(A_j\nabla F^n)(X^n(\mathbf{x}))$$
$$-\frac{1}{2}(l_j\, F^{n+1})(\mathbf{x}) + \frac{1}{2}(l_j\, F^n)(X^n(\mathbf{x})) - \widetilde{\lambda}\int_{y_{min}}^{y_{max}} \bar{F}^n(\bar{x}_1 + y, x_2)\nu(y)dy \quad = \quad 0$$

where $F = I, CI, V$ and $\bar{F}^n(\bar{x}_1 + y, x_2) = F^n(e^{\bar{x}_1+y}, x_2)$

Nonclassical Green formula

Let X:$\bar{\Omega} \to X(\bar{\Omega})$,X$\in \mathrm{C}^2(\bar{\Omega})$, be a vectorial inversible map. Let $\mathbf{F}_e$=∇X and assume that $\mathbf{F}_e{}^{-1} \in \mathrm{C}^1(\bar{\Omega})$. Then, we have:

$$\int_\Omega div\mathbf{w}(X(\mathbf{x}))\psi(\mathbf{x})dx = \int_\Gamma \mathbf{F}_e^{-T}(\mathbf{x})\mathbf{n}(\mathbf{x}) \cdot \mathbf{w}(X(\mathbf{x}))\psi(\mathbf{x})dA_x$$
$$- \int_\Omega \mathbf{F}_e^{-1}(\mathbf{x})\mathbf{w}(X(\mathbf{x})) \cdot \nabla\psi(\mathbf{x})dx$$
$$- \int_\Omega Div\mathbf{F}_e^{-T} \cdot \mathbf{w}(X(\mathbf{x}))\psi(\mathbf{x})dx,$$

where $\mathbf{w}\in \mathrm{H}^1(\mathrm{X}(\Omega))$ is a vectorial map and $\psi \in \mathrm{H}^1(\Omega)$ is a scalar function.

Some intermediate calculus: Green formulas (I)

- In the absence of jumps:

$$\int_\Omega \frac{F^{n+1} - F^n \circ X^n}{\Delta\tau_m}\psi d\boldsymbol{x} + \frac{1}{2}\int_\Omega A\nabla F^{n+1}\nabla\psi d\boldsymbol{x} +$$
$$+\frac{1}{2}\int_\Omega (\nabla X^n)^{-1}(A\nabla F^n)(X^n(\boldsymbol{x}))\nabla\psi d\boldsymbol{x} + \frac{1}{2}\int_\Omega (IF^{n+1})(\boldsymbol{x})\psi d\boldsymbol{x}$$
$$+\frac{1}{2}\int_\Omega (IF^n)(X^n(\boldsymbol{x}))\psi d\boldsymbol{x} + \int_\Omega Div((\nabla X^n)^{-T}(\mathbf{x}))(A\nabla F^n)(X^n(\mathbf{x}))\psi(\mathbf{x})d\mathbf{x}$$
$$= \frac{1}{2}\int_\Gamma \vec{n}\cdot A\nabla F^{n+1}\psi dA_{\boldsymbol{x}} + \frac{1}{2}\int_\Gamma (\nabla X^n)^{-T}\vec{n}\cdot(A\nabla F^n)(X^n(\boldsymbol{x}))\psi dA_{\boldsymbol{x}}$$

for $F = I, CI, V$

where $Div((\nabla X^n)^{-T}(\mathbf{x})) = \begin{pmatrix} 0 \\ \frac{r_\infty}{\kappa}(1 - exp(\kappa\Delta\tau_m)) \end{pmatrix}$

Some intermediate calculus: Green formulas (and II)

- In the presence of jumps:

$$\int_\Omega \frac{F^{n+1} - F^n \circ X^n}{\Delta\tau_m}\psi d\boldsymbol{x} + \frac{1}{2}\int_\Omega A_j \nabla F^{n+1} \nabla\psi d\boldsymbol{x} +$$

$$+ \frac{1}{2}\int_\Omega (\nabla X^n)^{-1}(A_j \nabla F^n)(X^n(\boldsymbol{x}))\nabla\psi d\boldsymbol{x} + \frac{1}{2}\int_\Omega (l_j F^{n+1})(\boldsymbol{x})\psi d\boldsymbol{x}$$

$$+ \frac{1}{2}\int_\Omega (l_j F^n)(X^n(\boldsymbol{x}))\psi d\boldsymbol{x} + \int_\Omega Div((\nabla X^n)^{-T}(\mathbf{x}))(A_j \nabla F^n)(X^n(\mathbf{x}))\psi(\mathbf{x})d\mathbf{x}$$

$$= \frac{1}{2}\int_\Gamma \vec{n}\cdot A_j \nabla F^{n+1}\psi dA_{\boldsymbol{x}} + \frac{1}{2}\int_\Gamma (\nabla X^n)^{-T}\vec{n}\cdot(A_j \nabla F^n)(X^n(\boldsymbol{x}))\psi dA_{\boldsymbol{x}}$$

$$+ \widetilde{\lambda}\int_\Omega \int_{y_{min}}^{y_{max}} \bar{F}^n(\bar{x}_1 + y, x_2)\nu(y)dy\psi d\mathbf{x} \quad \text{for} \quad F = I, CI, V$$

where $Div((\nabla X^n)^{-T}(\mathbf{x})) = \begin{pmatrix} 0 \\ \frac{r_\infty}{\kappa}(1 - exp(\kappa\Delta\tau_m)) \end{pmatrix}$ and $A_j = A$

Some intermediate calculus: Boundary terms (I)

- We have $\vec{n} \cdot A \nabla F^{n+1} = 0$ on $\Gamma_1^- \cup \Gamma_2^-$
- We impose $\frac{\partial F}{\partial x_1} = g_1 = 0$ on Γ_1^+ and $\frac{\partial F}{\partial x_2} = g_2 = 0$ on Γ_2^+ . So, we have:

$$\int_\Gamma \vec{n} \cdot A \nabla F^{n+1} \psi \, dA_{\mathbf{x}} = 0 \quad \text{for} \quad F = I, CI, V$$

Some intermediate calculus: Boundary terms (and II)

$$\int_\Gamma (\nabla X^n)^{-T} \vec{n} \cdot (A \nabla F^n)(X^n(\boldsymbol{x}))\psi dA_{\boldsymbol{x}} = \int_\Gamma \widetilde{g}^n \psi dA_{\boldsymbol{x}}$$

where $\widetilde{g}^n$ is defined as follows

$$\widetilde{g}^n(\mathbf{x}) = \begin{cases} -\left[(\nabla X^n)^{-T}\right]_{21}(\mathbf{x}) a_{22}(X_e^n(\mathbf{x})) \frac{\partial F}{\partial x_2}(X^n(\mathbf{x})) & on \quad \Gamma_1^- \\ 0 & on \quad \Gamma_2^- \\ \left[(\nabla X^n)^{-T}\right]_{22}(\mathbf{x}) a_{22}(X^n(\mathbf{x})) g_2^n(X^n(\mathbf{x})) & on \quad \Gamma_2^+ \\ \left[(\nabla X^n)^{-T}\right]_{11}(\mathbf{x}) a_{11}(X^n(\mathbf{x})) g_1^n(X^n(\mathbf{x})) \\ + \left[(\nabla X^n)^{-T}\right]_{21}(\mathbf{x}) a_{22}(X^n(\mathbf{x})) \frac{\partial F}{\partial x_2}(X^n(\mathbf{x})) & on \quad \Gamma_1^+ \end{cases}$$

Variational formulation in the absence of jumps

Find $F^{n+1} \in H^1(\Omega)$ such that, for all $\psi \in H^1(\Omega)$:

$$\int_\Omega F^{n+1}(\mathbf{x})\psi(\mathbf{x})d\mathbf{x} + \frac{\Delta\tau_m}{2}\int_\Omega (A\nabla F^{n+1})(\mathbf{x})\nabla\psi(\mathbf{x})d\mathbf{x}$$
$$+\frac{\Delta\tau_m}{2}\int_\Omega IF^{n+1}(\mathbf{x})\psi(\mathbf{x})d\mathbf{x} = \int_\Omega F^n(X^n(\mathbf{x}))\psi(\mathbf{x})d\mathbf{x}$$
$$-\frac{\Delta\tau_m}{2}\int_\Omega (\nabla X^n)^{-1}(\mathbf{x})(A\nabla F^n)(X^n(\mathbf{x}))\nabla\psi(\mathbf{x})d\mathbf{x}$$
$$-\frac{\Delta\tau_m}{2}\int_\Omega IF^n(X^n(\mathbf{x}))\psi(\mathbf{x})d\mathbf{x} + \frac{\Delta\tau_m}{2}\int_\Gamma \widetilde{g}^n(\mathbf{x})\psi(\mathbf{x})dA_\mathbf{x}$$
$$-\frac{\Delta\tau_m}{2}\int_\Omega Div((\nabla X^n)^{-T}(\mathbf{x}))(A\nabla F^n)(X^n(\mathbf{x}))\psi(\mathbf{x})d\mathbf{x}$$

Variational formulation in the presence of jumps

Find $F^{n+1} \in H^1(\Omega)$ such that, for all $\psi \in H^1(\Omega)$:

$$\int_\Omega F^{n+1}(\mathbf{x})\psi(\mathbf{x})d\mathbf{x} + \frac{\Delta\tau_m}{2}\int_\Omega (A\nabla F^{n+1})(\mathbf{x})\nabla\psi(\mathbf{x})d\mathbf{x}$$
$$+\frac{\Delta\tau_m}{2}\int_\Omega l_j F^{n+1}(\mathbf{x})\psi(\mathbf{x})d\mathbf{x} = \int_\Omega F^n(X^n(\mathbf{x}))\psi(\mathbf{x})d\mathbf{x}$$
$$-\frac{\Delta\tau_m}{2}\int_\Omega (\nabla X^n)^{-1}(\mathbf{x})(A\nabla F^n)(X^n(\mathbf{x}))\nabla\psi(\mathbf{x})d\mathbf{x}$$
$$-\frac{\Delta\tau_m}{2}\int_\Omega l_j F^n(X^n(\mathbf{x}))\psi(\mathbf{x})d\mathbf{x} + \frac{\Delta\tau_m}{2}\int_\Gamma \widetilde{g}^n(\mathbf{x})\psi(\mathbf{x})dA_\mathbf{x}$$
$$-\frac{\Delta\tau_m}{2}\int_\Omega Div((\nabla X^n)^{-T}(\mathbf{x}))(A\nabla F^n)(X^n(\mathbf{x}))\psi(\mathbf{x})d\mathbf{x}$$
$$+\Delta\tau_m\widetilde{\lambda}\int_\Omega\int_{y_{min}}^{y_{max}} \bar{F}^n(\bar{x}_1+y,x_2)\nu(y)dy\psi(\mathbf{x})d\mathbf{x}$$

Spatial Discretization: finite elements

For $F_h^0 \in F_h$, find $F_h = \{F_h^n\}_{n=1}^N \in [F_h]^N$ such that

$$\frac{1}{\Delta\tau_m} < \mathcal{D}_E^{n+1}[F], \psi_h > + < \mathcal{M}^n[F], \psi_h > = < \mathcal{N}^n, \psi_h >$$

for all $\psi_h \in F_h$ and $n = 0, ..., N-1$,

$$F_h = \{\phi_h \in \mathcal{C}^0(\overline{\Omega}) : \phi_h|_T \in \mathcal{Q}_2, \forall T \in \tau_h\}$$

Integral term approximation

Composite trapezoidal rule with m+1 points

$$\int_{y_{min}}^{y_{max}} \bar{F}^n(\bar{x}_1 + y, x_2)\nu(y)dy \approx \frac{h}{2}\Big[\bar{F}^n(\bar{x}_1 + y_{min}, x_2)\nu(y_{min})$$
$$+\bar{F}^n(\bar{x}_1 + y_{max}, x_2)\nu(y_{max}) + 2\sum_{j=1}^{m-1} \bar{F}^n(\bar{x}_1 + k_j, x_2)\nu(k_j)\Big]$$

where $k_j = y_{min} + jh$ for $j = 1, ..., m-1$ and $h = \frac{y_{max} - y_{min}}{m}$.

Augmented Lagrangian Active Set (ALAS) algorithm

Notation:

$$N := 1, 2, \ldots, N_{dof}, \quad N_{dof} := \text{dof of FEM}$$

For each time τ_m^n:

Find V_h^n, P_h^n and a decomposition $N = J^n \cup I^n$ such that

$$\begin{aligned} M_h V_h^n + P_h^n &= b_h^{n-1}, \\ [P_h^n]_j + \beta\,[V_h^n - TD]_j &> 0 \qquad \forall j \in J^n, \\ [P_h^n]_i &= 0 \qquad \forall i \in I^n, \end{aligned}$$

for any $\beta > 0$.

- I^n: discrete inactive set (non prepayment region) at time τ_m^n
- J^n: discrete active set (prepayment region) at time τ_m^n

Augmented Lagrangian Active Set (ALAS) algorithm

Notation:

$$N := 1, 2, \ldots, N_{dof}, \quad N_{dof} := \text{dof of FEM}$$

For each time τ_m^n:

Building sequences

$$\begin{aligned} V_{h,k}^n &\longrightarrow V_h^n \\ P_{h,k}^n &\longrightarrow P_h^n \\ I_k^n &\longrightarrow I^n \\ J_k^n &\longrightarrow J^n \end{aligned}$$

- I^n: discrete inactive set (non prepayment region) at time τ_m^n
- J^n: discrete active set (prepayment region) at time τ_m^n

Augmented Lagrangian Active Set (ALAS) algorithm

1. Set $V_{h,0}^n = TD_h^n$, $P_{h,0}^n = \max(b_h^n - M_h V_{h,0}^n, 0) \geq 0$, $\beta > 0$, $k = 0$.
2. Compute

$$\begin{aligned} Q_{h,k}^n &= \max\left\{0, P_{h,k}^n + \beta\left(V_{h,k}^n - TD_{h,k}^n\right)\right\}, \\ J_k^n &= \left\{j \in N, \left[Q_{h,k}^n\right]_j > 0\right\}, \\ I_k^n &= \{i \in N, \left[Q_{h,k}^n\right]_i = 0\}. \end{aligned}$$

3. If $k \geq 1$ and $J_k^n = J_{k-1}^n$ then convergence and stop.
4. Let V and P be the solution of the linear system (reduced)

$$\begin{aligned} [M_h]_{II}\,[V]_I &= \left[b^{n-1}\right]_I - [M_h]_{IJ}\,[TD]_J\,, \\ [V]_J &= [TD]_J\,, \\ P &= b^{n-1} - M_h V, \end{aligned}$$

for $I = I_k^n$ and $J = J_k^n$. Go to 2.