PDE models for pricing fixed rate mortgages and their insurance and coinsurance

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Outline

1. Fixed rate mortgage contract. Objectives
2. Mathematical models
   - PDE model
   - PIDE model
   - Additional mortgage characteristics
3. Numerical methods for PDE and PIDE models
4. Numerical results
5. Conclusions
Outline

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A **mortgage** is a financial contract in which the borrower obtains funds usually from a bank or a financial institution using the house as a collateral.

Some characteristics of the contract:

- **Interest rate:**
  - Fixed-Rate Mortgages (FRM).
  - Adjustable-Rate Mortgages (ARM).

- The term of the loan.

- Amount and frequency of payments: monthly payments.

- Prepayment option for the borrower under a penalization cost.

- Default option: insurance on the loan seems suitable.
Fixed rates does not mean this...

“You have been approved for a fixed-rate mortgage. That means if interest rates go up again and you’re not paying enough, we’ll fix it.”
Aim of the work

- Pricing of a mortgage with prepayment and default options.
- Obtain the value of the insurance and the coinsurance.
- Determine the fixed interest rate as an equilibrium rate.
- Prepayment can occur at any time during the life of the loan.
- Default happens only at monthly payment dates.
- PDE modelling approach: one PDE problem for each month.
- Prepayment option as a free boundary problem (one for each month): optimal prepayment boundary.
- Design appropriate numerical methods for solving the models.
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Underlying stochastic factors (I)

Evolution of the house value: geometric Brownian motion

\[ dH_t = (\mu - \delta)H_t dt + \sigma_H H_t dX^H_t \]

where

- \( \mu \): the instantaneous average rate of house-price appreciation,
- \( \delta \): the 'dividend-type' per unit service flow provided by the house,
- \( \sigma_H \): the house-price volatility
- \( X^H_t \): the standardized Wiener process for house price.

Equivalently, under the risk neutral measure the process satisfies

\[ dH_t = (r_t - \delta)H_t dt + \sigma_H H_t dX^H_t \]
Underlying stochastic factors (II)

Evolution of interest rate: CIR mean-reverting square root process

\[ dr_t = \kappa (\theta - r_t) dt + \sigma_r \sqrt{r_t} dX_t^r \]

where

- \( \kappa \): the speed of adjustment in the mean reverting process,
- \( \theta \): the long term mean of the short-term interest rate \( r_t \),
- \( \sigma_r \): the interest-rate volatility
- \( X_t^r \): the standardized Wiener process for interest rate.

Correlated Wiener processes

\[ dX_t^H dX_t^r = \rho dt \]
Dynamics of any asset (derivative) depending on house prices and interest rate

- \( F_t = F(t, H_t, r_t) \): price of any asset whose value depends on the house price \( H_t \), interest rate \( r_t \) and time \( t \).
- Applying Itô’s Lemma and suppressing the dependence on \( t \) for simplicity, the dynamics of \( F_t \) satisfies:

\[
dF = \frac{\partial F}{\partial t} dt + \frac{\partial F}{\partial H} dH + \frac{\partial F}{\partial r} dr \\
+ \frac{1}{2} \left( \sigma_H^2 H^2 \frac{\partial^2 F}{\partial H^2} + 2 \rho \sigma_H \sigma_r H \sqrt{r} \frac{\partial^2 F}{\partial H \partial r} + \sigma_r^2 r \frac{\partial^2 F}{\partial r^2} \right) dt
\]
PDE for any asset depending on house price and interest rate

Let $\Omega = (0, +\infty)^2$

**PDE in $(0, T) \times \Omega$**

\[
\frac{\partial F}{\partial t} + \frac{1}{2} \sigma_H^2 H^2 \frac{\partial^2 F}{\partial H^2} + \rho\sigma_H\sigma_r H \sqrt{r} \frac{\partial^2 F}{\partial H \partial r} + \frac{1}{2} \sigma_r^2 r \frac{\partial^2 F}{\partial r^2} + (r - \delta) H \frac{\partial F}{\partial H} + \kappa(\theta - r) \frac{\partial F}{\partial r} - rF = 0
\]

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Is geometric Brownian motion suitable for house prices evolution in certain market conditions?

**Figure:** U.S. Time Series of new home price returns for single family
Motivation of jump-diffusion models (I)

- The assumption of geometric Brownian motion for house price results reasonable under relatively stable market dynamics.
- However, for example time series of monthly house prices from January 1986 to June 2008 show several situations of jumps in house prices.
- Subprime crisis caused significantly downward jumps since November 2007.
- Other abnormal shocks are "Black Wendesday" in September 1992 or "Irak disarmament crisis" in July 1993, when U.S. Federal Reserve adapted an expansionary monetary policy.
Motivation of jump-diffusion models (II)


- The authors estimate parameters of a general jump-diffusion model using expectation maximum (EM) gradient algorithms based on U.S. housing price data.

- Empirical results show the likelihood ratio test (LRT) rejects the model without jumps at significance level of 99 percent for new home prices, although not in the case of second-hand houses.

- Thus, actually the house price evolution contains two parts: continuous diffusion and discontinuous jumps.
Stochastic model for the house value with jumps (I)

Evolution of the logarithmic house value, \( Z_t = \ln(H_t) \) with jumps

\[
dZ_t = \left( r_t - \frac{\sigma^2_H}{2} - \delta - \tilde{\lambda}\tilde{\kappa} \right) dt + \sigma_H dX_t^H + d \left( \sum_{i=1}^{N_t} V_i \right)
\]

where

- \( \delta \): the 'dividend-type' per unit service flow provided by the house,
- \( \sigma_H \): the house-price volatility,
- \( X_t^H \): the standardised Wiener process for house price,
- \( (N_t)_{t \geq 0} \): a Poisson process with parameter \( \tilde{\lambda} \)
- \( (V_i) \): a sequence of i.i.d random variables, with \( \tilde{\kappa} = E(\exp(V_i) - 1) \).
Stochastic model for the house value with jumps (and II)

Merton jump-diffusion model (1976)

- \((V_i) \sim N(\mu_j, \gamma_j^2)\) with the density

\[
\nu_m(y) = \frac{1}{\gamma_j \sqrt{2\pi}} \exp \left( -\frac{(y - \mu_j)^2}{2\gamma_j^2} \right)
\]

Kou jump-diffusion model (2002)

- \((V_i) \sim\) double-exponential distribution with the density

\[
\nu_k(y) = \begin{cases} 
q \alpha_2 \exp(\alpha_2 y), & y < 0 \\
p \alpha_1 \exp(-\alpha_1 y), & y \geq 0
\end{cases}
\]

with \(p, q, \alpha_1\) and \(\alpha_2\) positive constants with \(p + q = 1\) and \(\alpha_1 > 1\)
PIDE for any asset depending on house price and interest rate (I)

Evolution of the house value $H_t$ with jumps

$$dH_t = (r_t - \delta - \tilde{\lambda}\tilde{\kappa})H_t dt + \sigma_H dX_t^H + H_t d\left(\sum_{i=1}^{N_t}(Y_i - 1)\right)$$

where $Y_i = \exp(V_i)$.

PIDE in $(0, T) \times \Omega$

$$\frac{\partial F}{\partial t} + \frac{1}{2}\sigma_H^2 H^2 \frac{\partial^2 F}{\partial H^2} + \rho\sigma_H\sigma_r H\sqrt{r} \frac{\partial^2 F}{\partial H \partial r} + \frac{1}{2}\sigma_r^2 r \frac{\partial^2 F}{\partial r^2} + (r - \delta)H \frac{\partial F}{\partial H} + \kappa(\theta - r) \frac{\partial F}{\partial r} - rF$$

$$+ \int_{-\infty}^{\infty} \tilde{\lambda} \left[ F(t, H\exp(y), r) - F(t, H, r) - H(\exp(y) - 1) \frac{\partial F(t, H, r)}{\partial H} \right] \nu(y) dy = 0$$

$\nu(y) = \nu_m(y)$ for Merton model and $\nu(y) = \nu_k(y)$ for Kou one
PIDE for any asset depending on house price and interest rate (and II)

By considering that:

\[
\int_{-\infty}^{\infty} \nu(y) dy = 1, \quad \int_{-\infty}^{\infty} \exp(y) \nu_m(y) dy = e^{\mu_j + \gamma_j^2/2}, \quad \int_{-\infty}^{\infty} \exp(y) \nu_k(y) dy = \frac{p\alpha_1}{\alpha_1 - 1} + \frac{q\alpha_2}{\alpha_2 + 1}
\]

where \( \tilde{\kappa} = e^{\mu_j + \gamma_j^2/2} - 1 \) (Merton) or \( \tilde{\kappa} = \frac{p\alpha_1}{\alpha_1 - 1} + \frac{q\alpha_2}{\alpha_2 + 1} - 1 \) (Kou)
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Mortgage contract characteristics (I)

- Payments dates $T_m, m = 1, \ldots, M$
- Mortgage payment $MP$:
  \[
  MP = \frac{(c/12)(1 + c/12)^M P(0)}{(1 + c/12)^M - 1}
  \]
- Outstanding balance $P$:
  \[
  P(m - 1) = \frac{((1 + c/12)^M - (1 + c/12)^{m-1})P(0)}{(1 + c/12)^M - 1}
  \]

where

- $c$ is the fixed yearly contract rate,
- $P(0)$ is the initial amount loaned to the borrower
- $M$ is the term of the loan in months
Mortgage contract characteristics (and II)

- $\Delta T_m = T_m - T_{m-1}$: total time in month $m$
- $t_m = t - T_{m-1}$: time elapsed at month $m$
- $\tau_m = \Delta T_m - t_m$: time until the payment date in month $m$

Total debt payment $TD$ in case of early prepayment:

$$TD(\tau_m) = (1 + \Psi)(1 + c(\Delta T_m - \tau_m))P(m - 1)$$

where $\Psi$ is the prepayment penalty
IVP in the unbounded domain (I)

**PDE in (0, ΔT_m) × Ω**

\[-\frac{\partial F}{\partial \tau_m} + \frac{1}{2} \sigma_H^2 H^2 \frac{\partial^2 F}{\partial H^2} + \rho \sigma_H \sigma_r H \sqrt{r} \frac{\partial^2 F}{\partial H \partial r} + \frac{1}{2} \sigma_r^2 r \frac{\partial^2 F}{\partial r^2} + (r - \delta) H \frac{\partial F}{\partial H} + \kappa (\theta - r) \frac{\partial F}{\partial r} - rF = 0\]

**PIDE in (0, ΔT_m) × Ω**

\[-\frac{\partial F}{\partial \tau_m} + \frac{1}{2} \sigma_H^2 H^2 \frac{\partial^2 F}{\partial H^2} + \rho \sigma_H \sigma_r H \sqrt{r} \frac{\partial^2 F}{\partial H \partial r} + \frac{1}{2} \sigma_r^2 r \frac{\partial^2 F}{\partial r^2} + (r - \delta - \tilde{\lambda}) H \frac{\partial F}{\partial H} + \kappa (\theta - r) \frac{\partial F}{\partial r} - (r + \tilde{\lambda}) F + \tilde{\lambda} \int_{-\infty}^{\infty} F(t, H \exp(y), r) \nu(y) dy = 0\]
Contract value \( F(\tau_m, H, r) = V(\tau_m, H, r) \):

- at maturity
  \[
  V(\tau_M = 0, H, r) = \min(MP, H)
  \]

- at other payment dates \((1 \leq m \leq M - 1)\):
  \[
  V(\tau_m = 0, H, r) = \min(V(\tau_{m+1} = T_{m+1}, H, r) + MP, H)
  \]
Insurance value \( F(\tau_m, H, r) = I(\tau_m, H, r) \):

- at maturity

\[
I(\tau_M = 0, H, r) = \begin{cases} 
\min(\gamma(MP - H), \Gamma) & \text{(Default)} \\
0 & \text{(No default)}
\end{cases}
\]

- at other payment dates \( (1 \leq m \leq M - 1) \):

\[
I(\tau_m = 0, H, r) = \begin{cases} 
\min(\gamma[TD(\tau_m = 0) - H], \Gamma) & \text{(Default)} \\
I(\tau_{m+1} = \Delta T_{m+1}, H, r) & \text{(No default)}
\end{cases}
\]

where \( \gamma \) is a fraction of the total loss and \( \Gamma \) is the maximum indemnity.
IVP in the unbounded domain (and IV)

- Coinsurance value \( F(\tau_m, H, r) = Cl(\tau_m, H, r) \):
  - at maturity

\[
Cl(\tau_M = 0, H, r) = \begin{cases} 
\max((1 - \gamma)(MP - H), (MP - H) - \Gamma) & \text{(Default)} \\
0 & \text{(No default)} 
\end{cases}
\]

- at other payment dates \((1 \leq m \leq M - 1)\):

\[
Cl(\tau_m = 0, H, r) = \begin{cases} 
\max((1 - \gamma)[TD(\tau_m = 0) - H], [TD(\tau_m = 0) - H] - \Gamma) & \text{(Default)} \\
Cl(\tau_{m+1} = \Delta T_{m+1}, H, r) & \text{(No default)} 
\end{cases}
\]
Equilibrium condition at origination:

\[ V(\tau_1 = \Delta T_1, H_{initial}, r_{initial}; \Psi, c) + I(\tau_1 = \Delta T_1, H_{initial}, r_{initial}; \Psi, c) = (1-\xi)P(0) \]

where

- \( \xi P(0) \) is the arrangement fee,
- \( \Psi \) is the prepayment penalty
- and the contract rate \( c \) is the only free parameter. Its value can be obtained by using a \textbf{variable secant method}.

Free boundary problem (I)

Let us consider the following operators:

\[
\mathcal{L}_1[V] = -\frac{\partial V}{\partial \tau_m} + \frac{1}{2} \sigma_H^2 H^2 \frac{\partial^2 V}{\partial H^2} + \rho \sigma_r \sigma_H H \sqrt{r} \frac{\partial^2 V}{H \partial r} + \frac{1}{2} \sigma_r^2 r \frac{\partial^2 V}{\partial r^2} + (r - \delta) H \frac{\partial V}{\partial H} + \kappa(\theta - r) \frac{\partial V}{\partial r} - rV
\]

\[
\mathcal{L}_2[V] = -\frac{\partial V}{\partial \tau_m} + \frac{1}{2} \sigma_H^2 H^2 \frac{\partial^2 V}{\partial H^2} + \rho \sigma_r \sigma_H H \sqrt{r} \frac{\partial^2 V}{H \partial r} + \frac{1}{2} \sigma_r^2 r \frac{\partial^2 V}{\partial r^2} + (r - \delta - \tilde{\lambda} \tilde{\kappa}) H \frac{\partial V}{\partial H} + \kappa(\theta - r) \frac{\partial V}{\partial r} - (r + \tilde{\lambda}) F
\]

\[
+ \tilde{\lambda} \int_{-\infty}^{\infty} V(t, H \exp(y), r) \nu(y) dy
\]
So, the free boundary problem associated with the valuation of the mortgage contract, can be reduced to the linear complementarity problem:

\[ \mathcal{L}[V] \leq 0, \quad (TD(\tau_m) - V(\tau_m, H, r)) \geq 0, \quad (\mathcal{L}[V])(TD(\tau_m) - V(\tau_m, H, r)) = 0 \]

where \( \mathcal{L}[V] = \mathcal{L}_1[V] \) or \( \mathcal{L}[V] = \mathcal{L}_2[V] \)
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Divergence form in the unbounded domain (I)

Let us consider $\rho = 0$

- In the absence of jumps

\[
\mathcal{L}_1[F] = \partial_{\tau_m} F - \text{Div}(A \nabla F) + \vec{v} \cdot \nabla F + lF
\]

\[
A(H, r) = \begin{pmatrix}
\frac{1}{2} \sigma_H^2 H^2 & 0 \\
0 & \frac{1}{2} \sigma_r^2 r
\end{pmatrix}
\]

\[
\vec{v}(H, r) = \begin{pmatrix}
(\sigma_H^2 - r + \delta)H \\
(\frac{1}{2} \sigma_r^2 - \kappa(\theta - r))
\end{pmatrix}
\]

\[
l(\tau_m, H, r) = r
\]
Divergence form in the unbounded domain (II)

- In the presence of jumps

\[
\mathcal{L}_2[F] = \partial_{\tau_m} F - \text{Div}(A_j \nabla F) + \vec{v}_j \cdot \nabla F + l_j F - \tilde{\lambda} \int_{-\infty}^{\infty} F(\tau_m, H \exp(y), r) \nu(y) dy
\]

\[
A_j(H, r) = \begin{pmatrix}
\frac{1}{2} \sigma_H^2 H^2 & 0 \\
0 & \frac{1}{2} \sigma_r^2 r
\end{pmatrix}
\]

\[
\vec{v}_j(H, r) = \begin{pmatrix}
(\sigma_H^2 - r + \delta + \tilde{\lambda} \tilde{\kappa})H \\
\frac{1}{2} \sigma_r^2 - \kappa (\theta - r)
\end{pmatrix}
\]

\[
l_j(\tau_m, H, r) = r + \tilde{\lambda}
\]
Divergence form in the unbounded domain (and III)

Insurance and coinsurance \((F = I, CI)\): Cauchy problem

\[
\begin{cases}
\mathcal{L}[F] = f, \text{ in } (0, \Delta T_m) \times \Omega, \\
+\text{appropriate initial conditions for each month}
\end{cases}
\]

Mortgage contract value \((F = V)\): complementarity problem

\[
\begin{cases}
\max\{\mathcal{L}[F] - f, F - TD\} = 0, \text{ in } (0, \Delta T_m) \times \Omega, \\
+\text{appropriate initial conditions for each month}
\end{cases}
\]

where

\[
\mathcal{L}[F] = \mathcal{L}_1[F] \quad \text{or} \quad \mathcal{L}[F] = \mathcal{L}_2[F] \quad \text{and} \quad f(\tau_m, H, r) = 0
\]
Main difficulties in the numerical solution

- Unbounded domain in house price and interest rate directions
  - Localization + boundary conditions

- Unbounded domain of integration in the integral term
  - Localization

- Diffusion matrix is degenerated
  - (convection dominated problem)
  - Higher order Lagrange/Galerkin methods

- Nonlinearities in the free boundary problem
  - ALAS algorithm
Main difficulties in the numerical solution

- Unbounded domain in house price and interest rate directions
  - Localization + boundary conditions
- Unbounded domain of integration in the integral term
  - Localization
- Diffusion matrix is degenerated
  - (convection dominated problem)
  - Higher order Lagrange/Galerkin methods
- Nonlinearities in the free boundary problem
  - ALAS algorithm
Some references for numerical techniques

- Higher order characteristics

- Spatial discretization:

- ALAS algorithm:
We make the change of spatial variables: \( x_1 = \frac{H}{H_\infty} \) and \( x_2 = \frac{r}{r_\infty} \). So, \( \Omega^* = (0, 1) \times (0, 1) \)

\[ \Gamma_i^- = \{(x_1, x_2) \in \partial \Omega^* \mid x_i = 0\}, \quad \Gamma_i^+ = \{(x_1, x_2) \in \partial \Omega^* \mid x_i = 1\}, \quad i = 1, 2 \]

- **Insurance and coinsurance** \((J = I, Cl)\)

Find \( J : [0, \Delta T_m] \times \Omega^* \rightarrow \mathbb{R} \) satisfying the PDE

\[
\frac{\partial J}{\partial \tau_m} - \text{Div}(A \nabla J) + \vec{v} \cdot \nabla J + lJ = f \quad \text{in} \ (0, \Delta T_m) \times \Omega^*
\]

- **Mortgage contract value** \((V)\)

Find \( V : [0, \Delta T_m] \times \Omega^* \rightarrow \mathbb{R} \) satisfying the PDE

\[
\frac{\partial V}{\partial \tau_m} - \text{Div}(A \nabla V) + \vec{v} \cdot \nabla V + lV + P = f \quad \text{in} \ (0, \Delta T_m) \times \Omega^*
\]

jointly with the complementarity conditions

\[
V \leq TD, \quad P \geq 0, \quad P(TD - V) = 0 \quad \text{in} \ (0, \Delta T_m) \times \Omega^*
\]
Formulation in the bounded domain without jumps (and II)

where

\[
A(x_1, x_2) = \begin{pmatrix}
\frac{1}{2} \sigma_H^2 x_1^2 & 0 \\
0 & \frac{1}{2} \sigma_r^2 \frac{x_2}{r_\infty}
\end{pmatrix}
\]

\[
\vec{v}(x_1, x_2) = \begin{pmatrix}
(\sigma_H^2 - x_2 r_\infty + \delta)x_1 \\
(\frac{1}{2} \sigma_r^2 - \kappa(\theta - x_2 r_\infty))/r_\infty
\end{pmatrix}
\]

\[
l(\tau_m, x_1, x_2) = x_2 r_\infty
\]

\[
f(\tau_m, x_1, x_2) = 0
\]
Integral term localization

Change of variable: $\bar{x}_1 = \log(x_1)$

$$\int_{-\infty}^{\infty} F(\tau_m, x_1 \exp(y), x_2) \nu(y) dy \approx \int_{y_{\min}}^{y_{\max}} \bar{F}(\tau_m, \bar{x}_1 + y, x_2) \nu(y) dy$$

where $\bar{F}(\tau_m, \bar{x}_1 + y, x_2) = F(\tau_m, \exp(\bar{x}_1 + y), x_2)$.

If we consider the discrete grid $0 = x_{10}, x_{11}, \cdots, x_{1q} = 1$ then $y_{\min} = \log(x_{11})$ and $y_{\max} = \log(x_{1q})$.

Formulation in the bounded domain with jumps (I)

- **Insurance and coinsurance** \((J = I, CI)\)
  Find \(J : [0, \Delta T_m] \times \Omega^* \rightarrow \mathbb{R}\) such that
  \[
  \frac{\partial J}{\partial \tau_m} - \text{Div}(A_j \nabla J) + \tilde{v}_j \cdot \nabla J + l_j J - \tilde{\lambda} \int_{y_{\min}}^{y_{\max}} J(\tau_m, \tilde{x}_1 + y, x_2) \nu(y) dy = f
  \]
  in \((0, \Delta T_m) \times \Omega^*\)

- **Mortgage contract value** \((V)\)
  Find \(V : [0, \Delta T_m] \times \Omega^* \rightarrow \mathbb{R}\) satisfying the partial differential equation
  \[
  \frac{\partial V}{\partial \tau_m} - \text{Div}(A_j \nabla V) + \tilde{v}_j \cdot \nabla V + l_j V - \tilde{\lambda} \int_{y_{\min}}^{y_{\max}} V(\tau_m, \tilde{x}_1 + y, x_2) \nu(y) dy + P = f
  \]
  in \((0, \Delta T_m) \times \Omega^*\)
  the complementarity conditions
  \[
  V \leq TD, \quad P \geq 0, \quad P(TD - V) = 0 \quad \text{in} \ (0, \Delta T_m) \times \Omega^*
  \]
where

\[ A_j(x_1, x_2) = \begin{pmatrix}
\frac{1}{2} \sigma_H^2 x_1^2 & 0 \\
0 & \frac{1}{2} \sigma_r^2 \frac{x_2}{r_\infty}
\end{pmatrix} \]

\[ \tilde{v}_j(x_1, x_2) = \begin{pmatrix}
(\sigma_H^2 - x_2 r_\infty + \delta + \tilde{\lambda} \tilde{\kappa}) x_1 \\
(\frac{1}{2} \sigma_r^2 - \kappa (\theta - x_2 r_\infty))/r_\infty
\end{pmatrix} \]

\[ l_j(\tau_m, x_1, x_2) = x_2 r_\infty + \tilde{\lambda} \]

\[ f(\tau_m, x_1, x_2) = 0 \]
Moreover, we consider in both cases the previous initial conditions and the following boundary conditions:

\[
\frac{\partial J}{\partial x_1} = 0, \quad \frac{\partial V}{\partial x_1} = 0 \quad \text{on} \ (0, \Delta T_m) \times \Gamma^+_1
\]

\[
\frac{\partial J}{\partial x_1} = 0, \quad \frac{\partial V}{\partial x_2} = 0 \quad \text{on} \ (0, \Delta T_m) \times \Gamma^+_2
\]

### Fixed parameters in the mortgage valuation model

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<thead>
<tr>
<th>Economic framework</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Steady state spot rate, $\theta$</td>
<td>10 %</td>
</tr>
<tr>
<td>Speed of reversion, $\kappa$</td>
<td>25 %</td>
</tr>
<tr>
<td>House service flow, $\delta$</td>
<td>7.5%</td>
</tr>
<tr>
<td>Correlation coefficient, $\rho$</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
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<tr>
<th>Contract specifications</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial value of the house, $H_{initial}$</td>
<td>100000€</td>
</tr>
<tr>
<td>Ratio of the loan to value</td>
<td>95 %</td>
</tr>
<tr>
<td>Initial estimate for contract rate, $c_0$</td>
<td>10%</td>
</tr>
<tr>
<td>Prepayment penalty, $\Psi$</td>
<td>5%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
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<th>Value</th>
</tr>
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<tbody>
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**Table:** Fixed parameters in the mortgage valuation model
## Parameters of numerical methods

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**Table:** Parameters of numerical methods
Contract rate, mortgage, insurance and coinsurance value for $\sigma_r = 5\%$, $\sigma_H = 5\%$ without jumps in the house value

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<th>Contract value $V$</th>
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Contract rate, mortgage, insurance and coinsurance value for $\sigma_r = 10\%$, $\sigma_H = 5\%$ without jumps in the house value

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Contract rate, mortgage, insurance and coinsurance value for $\sigma_r = 5\%, \sigma_H = 10\%$ without jumps in the house value

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<th>Loan (years)</th>
<th>spot rate $r(0)$</th>
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Mortgage value at origination without jumps in the house value

Figure: Mortgage value at origination when the arrangement fee is 0.5%, the early exercise penalty takes the value of 5%, the contract rate is 9.3969%, the interest rate volatility is 10%, the house price volatility is 5%, the maturity of the contract is 25 years and the spot rate is 8%
Insurance value at origination without jumps in the house value

Figure: Insurance value at origination when the arrangement fee is 0.5%, the early exercise penalty takes the value of 5%, the contract rate is 9.3969%, the interest rate volatility is 10%, the house price volatility is 5%, the maturity of the contract is 25 years and the spot rate is 8%
Coinsurance value at origination without jumps in the house value

Figure: Coinsurance value at origination when the arrangement fee is 0.5%, the early exercise penalty takes the value of 5%, the contract rate is 9.3969%, the interest rate volatility is 10%, the house price volatility is 5%, the maturity of the contract is 25 years and the spot rate is 8%
Optimal prepayment boundary at origination without jumps in the house value

**Figure:** Optimal prepayment boundary at origination when the arrangement fee is 0.5%, the early exercise penalty takes the value of 5%, the contract rate is 9.3969%, the interest rate volatility is 10%, the house price volatility is 5%, the maturity of the contract is 25 years and the spot rate is 8%.
Parameters in the jump-diffusion models

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<th>Value</th>
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**Table**: Parameters in the jump-diffusion models
## Contract rate, mortgage, insurance and coinsurance value for $\sigma_r = 5\%$, $\sigma_H = 5\%$

under Merton jump-diffusion model for the house value

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<th>spot rate r(0)</th>
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<th>Contract rate c</th>
<th>Contract value V</th>
<th>Insurance I</th>
<th>Coinsurance CI</th>
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Contract rate, mortgage, insurance and coinsurance value for $\sigma_r = 5\%$, $\sigma_H = 5\%$ under Kou jump-diffusion model for the house value

<table>
<thead>
<tr>
<th>Loan (years)</th>
<th>spot rate $\xi$</th>
<th>Contract rate $c$</th>
<th>Contract value $V$</th>
<th>Insurance $I$</th>
<th>Coinsurance $CI$</th>
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<tr>
<td>15</td>
<td>8%</td>
<td>0%</td>
<td>14.2355%</td>
<td>92090</td>
<td>2910</td>
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<td></td>
<td></td>
<td>0.5%</td>
<td>14.1191%</td>
<td>91647</td>
<td>2878</td>
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<td>1%</td>
<td>14.0045%</td>
<td>91202</td>
<td>2848</td>
</tr>
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<td>1.5%</td>
<td>13.8920%</td>
<td>90759</td>
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<td></td>
<td>0%</td>
<td>15.2618%</td>
<td>92404</td>
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<td>15.1339%</td>
<td>91949</td>
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<td>15.0078%</td>
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<td>16.2317%</td>
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<td>1.5%</td>
<td>15.9438%</td>
<td>91314</td>
<td>2261</td>
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</tbody>
</table>
Optimal prepayment boundary at origination under Merton jump-diffusion model

**Figure:** Optimal prepayment boundary at origination when the arrangement fee is 0.5%, the early exercise penalty takes the value of 5%, the contract rate is 14.5892%, the interest rate volatility is 10%, the house price volatility is 5%, the maturity of the contract is 25 years and the spot rate is 8%
Optimal prepayment boundary at origination under Kou jump-diffusion model

**Figure:** Optimal prepayment boundary at origination when the arrangement fee is 0.5%, the early exercise penalty takes the value of 5%, the contract rate is 14.3926%, the interest rate volatility is 10%, the house price volatility is 5%, the maturity of the contract is 25 years and the spot rate is 8%
Conclusions (I)

- A PDE model to obtain the value of a fixed-rate mortgage with the prepayment and default options without jumps in the house value is proposed.
- A PIDE model to obtain the value of a fixed-rate mortgage with the prepayment and default with jumps in the house value is proposed.
- We consider Merton and Kou jump-diffusion processes.
- The stochastic variables are the house price and the interest rate.
- Insurance and coinsurance: initial boundary value problem for each month.
Conclusions (and II)

- Prepayment option in the mortgage contract: a complementarity problem for each month.
- The equilibrium interest rate is adjusted by using an iterative method.
- The mathematical models are solved by using appropriate numerical methods and several numerical results are presented.

Thank you for your attention!
Dynamic hedging methodology (I)

We build a portfolio

$$\Pi = F_1 - \Delta_2 F_2 - \Delta_1 H$$

Initially, the variation of the portfolio value in $[t, t + dt]$ is given by

$$d\Pi = dF_1 - \Delta_2 dF_2 - \Delta_1 dH$$

However, due to the dividend yield $\delta$, the portfolio must change by an amount

$$-\delta H\Delta_1 dt$$

Thus, actually the change of the portfolio value during $[t, t + dt]$ is

$$d\Pi = dF_1 - \Delta_2 dF_2 - \Delta_1 (dH + \delta H dt)$$
Thus, \( \Pi \) turns out to be risk free if we choose:

\[
\begin{align*}
\Delta_2 &= \frac{\partial F_1/\partial r}{\partial F_2/\partial r} \\
\Delta_1 &= \frac{\partial F_1}{\partial H} - \Delta_2 \frac{\partial F_2}{\partial H}
\end{align*}
\]
For the previous choice of $\Delta_1$ and $\Delta_2$, we have:

\[
d\Pi = \left[ \frac{\partial F_1}{\partial t} + \frac{1}{2} \left( \sigma_H^2 H^2 \frac{\partial^2 F_1}{\partial H^2} + 2 \rho \sigma_H \sigma_r \sqrt{r} \frac{\partial^2 F_1}{\partial H \partial r} + \sigma_r^2 \frac{\partial^2 F_1}{\partial r^2} \right) - \delta H \frac{\partial F_1}{\partial H} \right] dt
\]

By using no-arbitrage assumption, we have:

\[
d\Pi = r\Pi dt
\]

Identifying both quantities we get...
Dynamic hedging methodology (IV)

\[
\frac{1}{\partial F_1/\partial r} \left( \frac{\partial F_1}{\partial t} + \frac{1}{2} \sigma_H^2 H^2 \frac{\partial^2 F_1}{\partial H^2} + \rho \sigma_H \sigma_r H \sqrt{r} \frac{\partial^2 F_1}{\partial H \partial r} + \frac{1}{2} \sigma_r^2 \frac{\partial^2 F_1}{\partial r^2} + (r - \delta)H \frac{\partial F_1}{\partial H} - rF_1 \right) =
\]

\[
\frac{1}{\partial F_2/\partial r} \left( \frac{\partial F_2}{\partial t} + \frac{1}{2} \sigma_H^2 H^2 \frac{\partial^2 F_2}{\partial H^2} + \rho \sigma_H \sigma_r H \sqrt{r} \frac{\partial^2 F_2}{\partial H \partial r} + \frac{1}{2} \sigma_r^2 \frac{\partial^2 F_2}{\partial r^2} + (r - \delta)H \frac{\partial F_2}{\partial H} - rF_2 \right)
\]
Dynamic hedging methodology (and V)

Both sides are independent of maturity. So,

\[
\frac{1}{\partial F/\partial r} \left( \frac{\partial F}{\partial t} + \frac{1}{2} \sigma_H^2 H^2 \frac{\partial^2 F}{\partial H^2} + \rho \sigma_H \sigma_r H \sqrt{r} \frac{\partial^2 F}{\partial H \partial r} + \frac{1}{2} \sigma_r^2 r \frac{\partial^2 F}{\partial r^2} + (r - \delta) H \frac{\partial F}{\partial H} - rF \right) = a(t, H, r)
\]

where \(a(t, H, r) = -\kappa(\theta - r)\).

J. A. Azevedo-Pereira, D. P. Newton and D. A. Paxson, UK Fixed Rate Repayment Mortgage and Mortgage Indemnity Valuation, Real Estate Economics, 30 (2002), 185-211
Time discretization: method of characteristics (I)

Vector associated with the first order derivative terms (velocity field):

- In the absence of jumps:
  \[ \vec{v}_1 = \vec{v} = \left( \frac{(\sigma_H^2 - x_2 r_\infty + \delta)x_1}{(\frac{1}{2} \sigma_r^2 - \kappa(\theta - x_2 r_\infty)) / r_\infty} \right) \]

- In the presence of jumps:
  \[ \vec{v}_1 = \vec{v}_j = \left( \frac{(\sigma_H^2 - x_2 r_\infty + \delta + \tilde{\lambda} \tilde{\kappa})x_1}{(\frac{1}{2} \sigma_r^2 - \kappa(\theta - x_2 r_\infty)) / r_\infty} \right) \]

Characteristic curves through \( x = (x_1, x_2) \) at time \( \tau_m^{n+1} \): \( X(x, \tau_m^{n+1}; s) \)

\[ \frac{\partial}{\partial s} X(x, \tau_m^{n+1}; s) = \vec{v}_1(X(x, \tau_m^{n+1}; s)), \quad X(x, \tau_m^{n+1}; \tau_m^{n+1}) = x \]
Time discretization: method of characteristics (II)

Characteristic curves can be exactly computed:
- In the absence of jumps

\[
X_1^n(x) = x_1 \exp\left(-\left(\sigma^2_H + \delta + \frac{\sigma_r^2}{2\kappa} - \theta\right)\Delta\tau_m\right) \times \\
\exp\left(\left(-\frac{x_2 r_\infty}{\kappa} - \frac{\sigma_r^2}{2\kappa^2} + \frac{\theta}{\kappa}\right)(\exp(-\kappa \Delta\tau_m) - 1)\right)
\]

\[
X_2^n(x) = \left(-\frac{\sigma_r^2}{2\kappa r_\infty} + \frac{\theta}{r_\infty}\right)(1 - \exp(-\kappa \Delta\tau_m)) + x_2 \exp(-\kappa \Delta\tau_m)
\]

where \(X^n(x) := X(x, \tau_{m+1}^n; \tau_m^n)\)

In the presence of jumps

\[ X_1^n(x) = x_1 \exp\left( -\left( \sigma_H^2 + \delta + \frac{\sigma_r^2}{2\kappa} - \theta + \tilde{\lambda}\tilde{\kappa} \right) \Delta \tau_m \right) \times \]

\[ \exp\left( \left( -\frac{x_2 r_\infty}{\kappa} - \frac{\sigma_r^2}{2\kappa^2} + \frac{\theta}{\kappa} \right) \left( \exp(-\kappa \Delta \tau_m) - 1 \right) \right) \]

\[ X_2^n(x) = \left( -\frac{\sigma_r^2}{2\kappa r_\infty} + \frac{\theta}{r_\infty} \right) \left( 1 - \exp(-\kappa \Delta \tau_m) \right) + x_2 \exp(-\kappa \Delta \tau_m) \]

where \( X^n(x) := X(x, \tau_{m+1}^n; \tau_m^n) \)

Crank-Nicolson-Characteristic (CN-char) scheme

Time step \( \Delta \tau_m = \frac{\Delta T_m}{N} \)

Time meshpoints \( \tau_m^n = n \Delta \tau_m, \ n = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots, N. \)

Characteristics for material derivative approximation:

\[
\frac{DF}{D\tau_m} = \frac{F^{n+1} - F^n \circ X^n}{\Delta \tau_m}
\]

where \( X^n(x) := X(x, \tau_m^{n+1}; \tau_m) \)

Crank-Nicolson around \( (X(x, \tau_m^{n+1}; \tau_m), \tau_m) \) for \( \tau = \tau_m^{n+\frac{1}{2}} \)
CN-char scheme for time discretization without jumps

For \( n=0,\ldots,N-1 \), find \( F^{n+1} \) such that:

\[
\frac{F^{n+1}(x) - F^n(X^n(x))}{\Delta \tau_m} - \frac{1}{2} \text{Div}(A \nabla F^{n+1})(x) - \frac{1}{2} \text{Div}(A \nabla F^n)(X^n(x)) \\
+ \frac{1}{2} (l F^{n+1})(x) + \frac{1}{2} (l F^n)(X^n(x)) = 0
\]

where \( F = I, CI, V \)
For $n=0,\ldots,N-1$, find $F^{n+1}$ such that:

$$\frac{F^{n+1}(x) - F^n(X^n(x))}{\Delta \tau_m} - \frac{1}{2} \text{Div}(A_j \nabla F^{n+1})(x) - \frac{1}{2} \text{Div}(A_j \nabla F^n)(X^n(x))$$

$$- \frac{1}{2} (l_j F^{n+1})(x) + \frac{1}{2} (l_j F^n)(X^n(x)) - \bar{\lambda} \int_{y_{\min}}^{y_{\max}} \bar{F}^n(\bar{x}_1 + y, x_2) \nu(y) dy = 0$$

where $F = I, Cl, V$ and $\bar{F}^n(\bar{x}_1 + y, x_2) = F^n(e^{\bar{x}_1+y}, x_2)$
Let $X: \Omega \rightarrow X(\bar{\Omega}), X \in C^2(\bar{\Omega})$, be a vectorial invertible map. Let $F_e = \nabla X$ and assume that $F_e^{-1} \in C^1(\bar{\Omega})$. Then, we have:

$$\int_{\Omega} \text{div} w(X(x)) \psi(x) dx = \int_{\Gamma} F_e^{-T}(x)n(x) \cdot w(X(x)) \psi(x) dA_x$$

$$\quad - \int_{\Omega} F_e^{-1}(x)w(X(x)) \cdot \nabla \psi(x) dx$$

$$\quad - \int_{\Omega} \text{Div} F_e^{-T} \cdot w(X(x)) \psi(x) dx,$$

where $w \in H^1(X(\Omega))$ is a vectorial map and $\psi \in H^1(\Omega)$ is a scalar function.
Some intermediate calculus: Green formulas (I)

- In the absence of jumps:

\[
\int_{\Omega} \frac{F^{n+1} - F^n \circ X^n}{\Delta \tau_m} \psi d\mathbf{x} + \frac{1}{2} \int_{\Omega} A \nabla F^{n+1} \nabla \psi d\mathbf{x} + \\
\frac{1}{2} \int_{\Omega} (\nabla X^n)^{-1} (A \nabla F^n)(X^n(\mathbf{x})) \nabla \psi d\mathbf{x} + \frac{1}{2} \int_{\Omega} (lF^{n+1})(\mathbf{x}) \psi d\mathbf{x} \\
+ \frac{1}{2} \int_{\Omega} (lF^n)(X^n(\mathbf{x})) \psi d\mathbf{x} + \int_{\Omega} \text{Div}((\nabla X^n)^{-T}(\mathbf{x}))(A \nabla F^n)(X^n(\mathbf{x})) \psi(\mathbf{x}) d\mathbf{x} \\
= \frac{1}{2} \int_{\Gamma} \mathbf{n} \cdot A \nabla F^{n+1} \psi d\mathbf{A}_{\mathbf{x}} + \frac{1}{2} \int_{\Gamma} (\nabla X^n)^{-T} \mathbf{n} \cdot (A \nabla F^n)(X^n(\mathbf{x})) \psi d\mathbf{A}_{\mathbf{x}}
\]

for \( F = I, CI, V \)

where \( \text{Div}((\nabla X^n)^{-T}(\mathbf{x})) = \begin{pmatrix} 0 \\ r_{\infty} \kappa (1 - \exp(\kappa \Delta \tau_m)) \end{pmatrix} \)
Some intermediate calculus: Green formulas (and II)

- In the presence of jumps:

\[
\int_{\Omega} \frac{F^{n+1} - F^n \circ X^n}{\Delta \tau_m} \psi d\mathbf{x} + \frac{1}{2} \int_{\Omega} A_j \nabla F^{n+1} \nabla \psi d\mathbf{x} + \\
\frac{1}{2} \int_{\Omega} (\nabla X^n)^{-1} (A_j \nabla F^n)(X^n(\mathbf{x})) \nabla \psi d\mathbf{x} + \frac{1}{2} \int_{\Omega} (l_j F^{n+1})(\mathbf{x}) \psi d\mathbf{x} \\
+ \frac{1}{2} \int_{\Omega} (l_j F^n)(X^n(\mathbf{x})) \psi d\mathbf{x} + \int_{\Omega} \text{Div}((\nabla X^n)^{-T}(\mathbf{x}))(A_j \nabla F^n)(X^n(\mathbf{x})) \psi(\mathbf{x}) d\mathbf{x} \\
= \frac{1}{2} \int_{\Gamma} \mathbf{n} \cdot A_j \nabla F^{n+1} \psi dA_{\mathbf{x}} + \frac{1}{2} \int_{\Gamma} (\nabla X^n)^{-T} \mathbf{n} \cdot (A_j \nabla F^n)(X^n(\mathbf{x})) \psi dA_{\mathbf{x}} \\
+ \tilde{\lambda} \int_{\Omega} \int_{y_{\min}}^{y_{\max}} F^n(\tilde{x}_1 + y, x_2) \nu(y) dy \psi d\mathbf{x} \quad \text{for} \quad F = I, CI, V
\]

where \( \text{Div}((\nabla X^n)^{-T}(\mathbf{x})) = \left( \begin{array}{cc} 0 \\ \frac{r_\infty}{\kappa} (1 - \exp(\kappa \Delta \tau_m)) \end{array} \right) \) and \( A_j = A \)
Some intermediate calculus: Boundary terms (I)

- We have \( \vec{n} \cdot A \nabla F^{n+1} = 0 \) on \( \Gamma_1^- \cup \Gamma_2^- \)
- We impose \( \frac{\partial F}{\partial x_1} = g_1 = 0 \) on \( \Gamma_1^+ \) and \( \frac{\partial F}{\partial x_2} = g_2 = 0 \) on \( \Gamma_2^+ \). So, we have:

\[
\int_{\Gamma} \vec{n} \cdot A \nabla F^{n+1} \psi dA_x = 0 \quad \text{for} \quad F = I, CI, V
\]
Some intermediate calculus: Boundary terms (and II)

\[ \int_\Gamma (\nabla X^n)^{-T} \mathbf{n} \cdot (A \nabla F^n)(X^n(x)) \psi dA_x = \int_\Gamma \tilde{g}^n \psi dA_x \]

where \( \tilde{g}^n \) is defined as follows

\[ \tilde{g}^n(x) = \begin{cases} 
- [ (\nabla X^n)^{-T} ]_{21} (x) a_{22}(X^n(x)) \frac{\partial F}{\partial x_2}(X^n(x)) & \text{on } \Gamma_1^- \\
0 & \text{on } \Gamma_2^- \\
[ (\nabla X^n)^{-T} ]_{22} (x) a_{22}(X^n(x)) g_2^n(X^n(x)) & \text{on } \Gamma_2^+ \\
[ (\nabla X^n)^{-T} ]_{11} (x) a_{11}(X^n(x)) g_1^n(X^n(x)) + [ (\nabla X^n)^{-T} ]_{21} (x) a_{22}(X^n(x)) \frac{\partial F}{\partial x_2}(X^n(x)) & \text{on } \Gamma_1^+ 
\end{cases} \]
Variational formulation in the absence of jumps

Find $F^{n+1} \in H^1(\Omega)$ such that, for all $\psi \in H^1(\Omega)$:

$$
\int_{\Omega} F^{n+1}(x)\psi(x)dx + \frac{\Delta \tau_m}{2} \int_{\Omega} (A \nabla F^{n+1})(x) \nabla \psi(x)dx \\
+ \frac{\Delta \tau_m}{2} \int_{\Omega} IF^{n+1}(x)\psi(x)dx = \int_{\Omega} F^n(X^n(x))\psi(x)dx \\
- \frac{\Delta \tau_m}{2} \int_{\Omega} (\nabla X^n)^{-1}(x)(A \nabla F^n)(X^n(x)) \nabla \psi(x)dx \\
- \frac{\Delta \tau_m}{2} \int_{\Omega} IF^n(X^n(x))\psi(x)dx + \frac{\Delta \tau_m}{2} \int_{\Gamma} \tilde{g}^n(x)\psi(x)dA_x \\
- \frac{\Delta \tau_m}{2} \int_{\Omega} \text{Div}((\nabla X^n)^{-T}(x))(A \nabla F^n)(X^n(x))\psi(x)dx
$$
Variational formulation in the presence of jumps

Find $F^{n+1} \in H^1(\Omega)$ such that, for all $\psi \in H^1(\Omega)$:

$$
\int_{\Omega} F^{n+1}(x) \psi(x) dx + \frac{\Delta \tau_m}{2} \int_{\Omega} (A \nabla F^{n+1})(x) \nabla \psi(x) dx
$$

$$
+ \frac{\Delta \tau_m}{2} \int_{\Omega} l_j F^{n+1}(x) \psi(x) dx = \int_{\Omega} F^n(X^n(x)) \psi(x) dx
$$

$$
- \frac{\Delta \tau_m}{2} \int_{\Omega} (\nabla X^n)^{-1}(x) (A \nabla F^n)(X^n(x)) \nabla \psi(x) dx
$$

$$
- \frac{\Delta \tau_m}{2} \int_{\Omega} l_j F^n(X^n(x)) \psi(x) dx + \frac{\Delta \tau_m}{2} \int_{\Gamma} \tilde{g}^n(x) \psi(x) dA_x
$$

$$
- \frac{\Delta \tau_m}{2} \int_{\Omega} \text{Div}((\nabla X^n)^{-T}(x)) (A \nabla F^n)(X^n(x)) \psi(x) dx
$$

$$
+ \Delta \tau_m \tilde{\lambda} \int_{\Omega} \int_{y \text{max}}^{y \text{max}} \bar{F}^n(\bar{x}_1 + y, x_2) \nu(y) dy \psi(x) dx
$$
Spatial Discretization: finite elements

For $F_h^0 \in F_h$, find $F_h = \{F_h^n\}_{n=1}^{N} \in [F_h]^N$ such that

$$\frac{1}{\Delta \tau_m} < D_E^{n+1}[F], \psi_h > + < M^n[F], \psi_h > = < N^n, \psi_h >$$

for all $\psi_h \in F_h$ and $n = 0, ..., N - 1$,

$$F_h = \{\phi_h \in C^0(\overline{\Omega}) : \phi_h|_T \in Q_2, \forall T \in \tau_h\}$$
Integral term approximation

Composite trapezoidal rule with \( m+1 \) points

\[
\int_{y_{\text{min}}}^{y_{\text{max}}} \bar{F}_n(\bar{x}_1 + y, x_2) \nu(y) dy \approx \frac{h}{2} \left[ \bar{F}_n(\bar{x}_1 + y_{\text{min}}, x_2) \nu(y_{\text{min}}) + \bar{F}_n(\bar{x}_1 + y_{\text{max}}, x_2) \nu(y_{\text{max}}) + 2 \sum_{j=1}^{m-1} \bar{F}_n(\bar{x}_1 + k_j, x_2) \nu(k_j) \right]
\]

where \( k_j = y_{\text{min}} + jh \) for \( j = 1, \ldots, m - 1 \) and \( h = \frac{y_{\text{max}} - y_{\text{min}}}{m} \).
Augmented Lagrangian Active Set (ALAS) algorithm

Notation:

\[ N := 1, 2, \ldots, N_{\text{dof}}, \quad N_{\text{dof}} := \text{dof of FEM} \]

For each time \( \tau^n_m \):

Find \( V^n_h, P^n_h \) and a decomposition \( N = J^n \cup I^n \) such that

\[
M_h V^n_h + P^n_h = b^{n-1}_h,
\]

\[
[P^n_h]_j + \beta [V^n_h - TD]_j > 0 \quad \forall j \in J^n,
\]

\[
[P^n_h]_i = 0 \quad \forall i \in I^n,
\]

for any \( \beta > 0 \).

- \( I^n \): discrete inactive set (non prepayment region) at time \( \tau^n_m \)
- \( J^n \): discrete active set (prepayment region) at time \( \tau^n_m \)
Augmented Lagrangian Active Set (ALAS) algorithm

Notation:

\( N := 1, 2, \ldots, N_{dof} \), \( N_{dof} := \text{dof of FEM} \)

For each time \( \tau^n_m \):

Building sequences

\[
\begin{align*}
V_{h,k}^n & \longrightarrow V_h^n \\
P_{h,k}^n & \longrightarrow P_h^n \\
I_k^n & \longrightarrow I^n \\
J_k^n & \longrightarrow J^n \\
\end{align*}
\]

- \( I^n \): discrete inactive set (non prepayment region) at time \( \tau^n_m \)
- \( J^n \): discrete active set (prepayment region) at time \( \tau^n_m \)
Augmented Lagrangian Active Set (ALAS) algorithm

1. Set $V_{h,0}^n = TD_{h}^n$, $P_{h,0}^n = \max(b_h^n - M_h V_{h,0}^n, 0) \geq 0$, $\beta > 0$, $k = 0$. 
2. Compute
   
   \[
   Q_{h,k}^n = \max \left\{ 0, P_{h,k}^n + \beta \left( V_{h,k}^n - TD_{h,k}^n \right) \right\},
   \]
   
   \[
   J_k^n = \left\{ j \in N, \left[ Q_{h,k}^n \right]_j > 0 \right\},
   \]
   
   \[
   I_k^n = \left\{ i \in N, \left[ Q_{h,k}^n \right]_i = 0 \right\}.
   \]
3. If $k \geq 1$ and $J_k^n = J_{k-1}^n$ then convergence and stop.
4. Let $V$ and $P$ be the solution of the linear system (reduced)
   
   \[
   [M_h]_{ll} [V]_l = [b^{n-1}]_l - [M_h]_{lj} [TD]_j,
   \]
   
   \[
   [V]_J = [TD]_J,
   \]
   
   \[
   P = b^{n-1} - M_h V,
   \]
   
   for $l = I_k^n$ and $J = J_k^n$. Go to 2.