Empirical Approach to the Heston Model Parameters on the Exchange Rate USD / COP

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Outline

- Proposal: empirical calibration of Heston stochastic volatility model for USD / COP under physical measure of risk
- Parameter estimation: done by developing an algorithm that performs simulated trajectories for $USDCOP_{Heston}$ & matching pdf of simulated paths with pdf coming from the real exchange rate
- Calibration: two-sample KS test & Nelder Mead simplex direct search
- At the end: the results show that although achieving multiple optima parameter values -depending on an initial vector parameter - is posible, one of these could be chosen according to financial market information

Limitations of the Black-Scholes-Merton (BMS - 1973) in option valuation have been pointed out with rigour, e.g. volatility surfaces do not reflect the reality in markets.

Facts

- given a financial asset, the volatility process is nonnegative and shows mean reversion
- the model provides leptokurtic and fat tails returns distribution
- explains the smile and skew effect appearing in the real implied volatility
- it is analytically tractable in aspects of vanilla option valuation
- BSM(1973), Heston(1993), Gathered(2006), Fabrice(2013), Fatone(2014), Hurn(2015)

Seminal models in S.V: Hull & White (1987), Scott (1987), Stein & Stein (1991), and Heston (1993).
Heston model: driven by the Itô differential equations:

$$dS_t = \mu \ S_t dt + \sqrt{v_t} \ S_t dW_t^{(1)} \tag{1}$$

$$dv_t = \kappa \left(\theta - v_t\right) \ dt + \sigma \sqrt{v_t} dW_t^{(2)} \tag{2}$$

- μ, θ, κ and σ are constants
- S_t is conditioned to S_0, v_0, and $\{v_s, 0 \leq s \leq t\}$
- $Corr(dW_t^{(1)}, dW_t^{(2)}) = \rho dt$
- $\{v_t\} > 0$ if $2\kappa\theta \geq \sigma^2$

Heston under risk-neutral measure and PDE

- Risk neutral measure

$$dS_t = r\ S_t dt + \sqrt{v_t}\ S_t d\tilde{W}_t^{(1)}$$
$$v_t = \kappa^* \left(\theta^* - v_t\right)\ dt + \sigma\sqrt{v_t} d\tilde{W}_t^{(2)}$$

  where $\kappa^* = \kappa + \lambda$, $\theta^* = \kappa\theta/(\kappa + \lambda)$, and volatility risk premium $\lambda(S_t, v_t, t) = \lambda v_t$, λ constant

- Derivative price U is driven by the PDE

$$\frac{1}{2} v_t S_t^2 \frac{\partial^2 U}{\partial S_t^2} + \rho\sigma v_t S_t \frac{\partial^2 U}{\partial S_t \partial v_t} + \frac{1}{2}\sigma^2 v_t \frac{\partial^2 U}{\partial v_t^2} + r S_t \frac{\partial U}{\partial S_t} + \left(\kappa\left[\theta - v_t\right] - \lambda(S_t, v_t, t)\right) \frac{\partial U}{\partial v_t} - rU + \frac{\partial U}{\partial t} = 0$$

Heston call option formula

$$C(S_t, v_t, t) = S_t P_1 - K e^{-r\tau} P_2$$

$$P_j\left(x, v, T; \ln(K)\right) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\left[\frac{e^{-i\phi\ln(K)} f_j(x, v, t; \phi)}{i\phi}\right] d\phi$$

where

$$\begin{aligned}
f_j(x, v, T : \phi) &= e^{C(\tau;\ \phi) + D(\tau;\ \phi)v + i\phi x}, \\
C(\tau; \phi) &= r\phi i\tau + \frac{a}{\sigma^2}\left\{(b_j - \rho\sigma\phi i + d)\tau - 2\ln\left[\frac{1 - ge^{d\tau}}{1 - g}\right]\right\}, \\
D(\tau; \phi) &= \frac{b_j - \rho\sigma\phi i + d}{\sigma^2}\left[\frac{1 - e^{d\tau}}{1 - ge^{d\tau}}\right], \\
g &= \frac{b_j - \rho\sigma\phi i + d}{b_j - \rho\sigma\phi i - d}, \\
d &= \sqrt{\left(\rho\sigma\phi i - b_j\right)^2 - \sigma^2\left(2u_j\phi i - \phi^2\right)},
\end{aligned}$$

$$u_1 = 1/2,\ u_2 = -1/2,\ a = \kappa\theta,\ b_1 = \kappa + \lambda - \rho\sigma,\ b_2 = \kappa + \lambda$$

with $\lambda = \lambda(S_t, v_t, t) = k\sqrt{v}$ for a constant k .

The parameters were taken as $S = 100$, $r = q = \lambda = 0$, $\kappa = 10$, $\theta = v_0 = 0.05$, $\sigma = 0.5$, $\rho = -0.5$, $T \in (0, 0.25]$ and $K \in [50, 200]$

Figure 1: Case $j = 2$, Heston Integral for ATM call

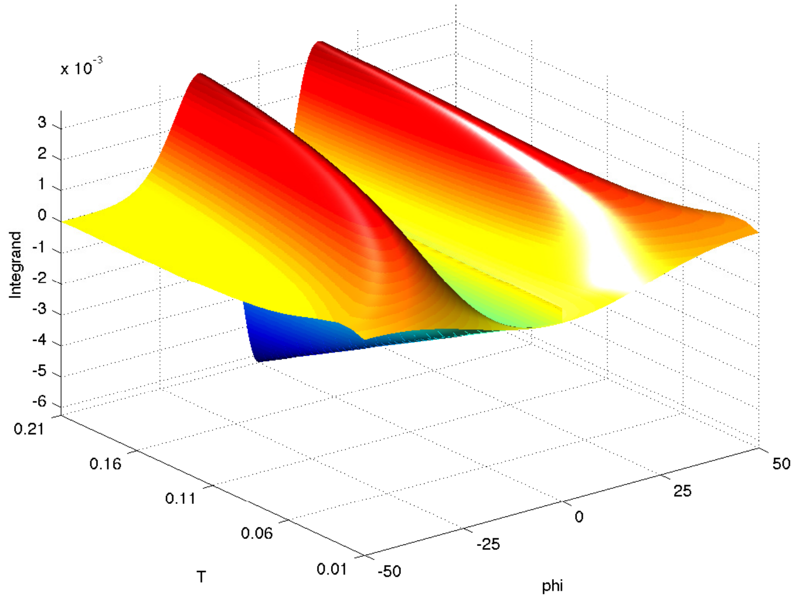

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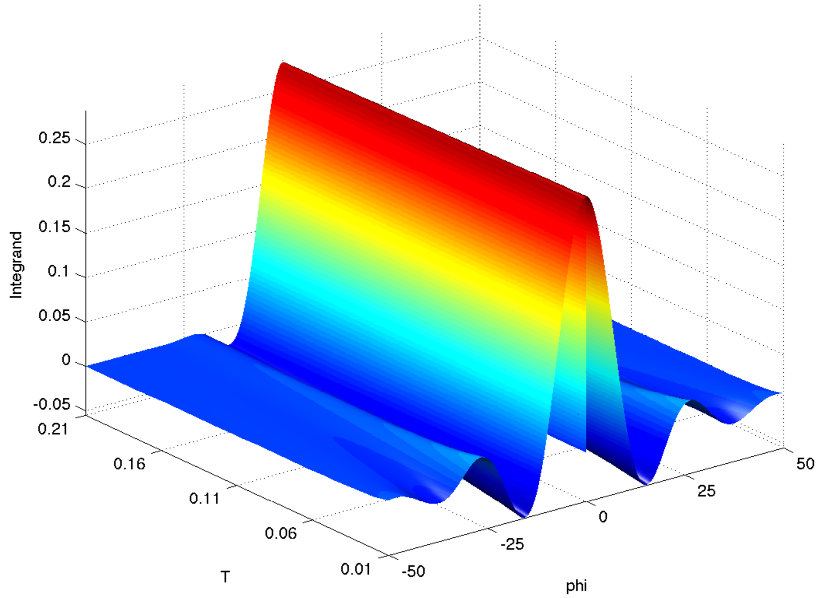
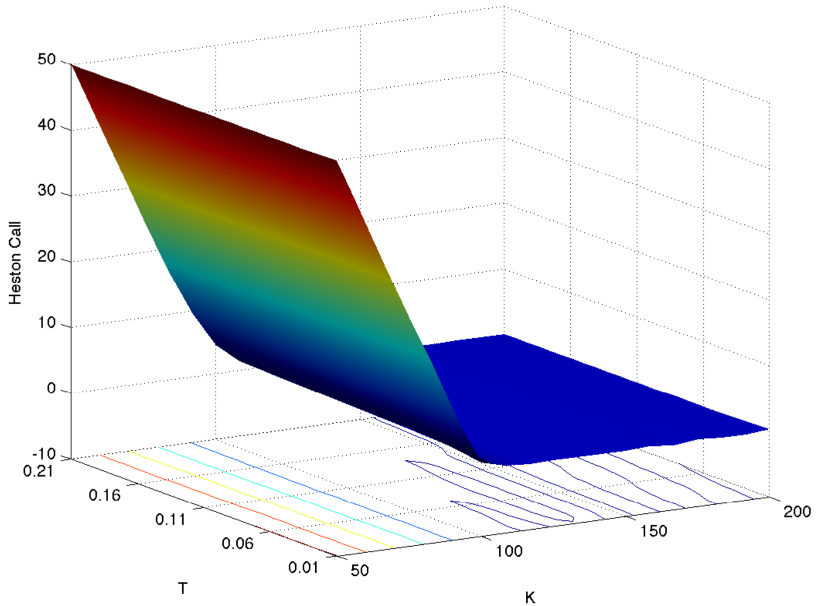
Figure 2: Case $j = 2$, Heston Integral for ITM call



Heston call surface on $K - T$ plane

The parameters were taken as $S = 100$, $r = q = \lambda = 0$, $\kappa = 10$, $\theta = v_0 = 0.05$, $\sigma = 0.5$, $\rho = -0.5$, $T \in (0, 0.25]$ and $K \in [50, 200]$

Figure 3: Case $j = 2$, Heston call $C(K, T)$



ρ: controls the skewness of the density of $\ln S_T \longrightarrow$ positive correlation implies an increase in variance for bullish markets and the contrary effect happens for negative correlation.

Table 1: Call price relation in BSM and Heston model for different ρ

	$\rho > 0$	$\rho < 0$
OTM European Call	$Heston > BSM$	$Heston < BSM$
ITM European Call	$Heston < BSM$	$Heston > BSM$

Risk C-price sensitivities

σ: affects the kurtosis of the distribution of the $\ln S_T$ and the curvature of the smile produced by the implied volatility of Heston model.

kurtosis and curvature increase as sigma increases $\longrightarrow$ fat tails in the distribution of $\ln S_T$.

So in ITM and OTM options: we have $C_{Heston} > C_{BSM}$
in ATM options: we have $C_{Heston} < C_{BSM}$
and such relations are kept in a deeply way as sigma increases.

Parameters calibration - some references

- Methods of Moments like SMM (*Simulated Method of Moments*), GMM (*Generalized Method of Moments*) and EMM (*Efficient Method of Moments*) in Duffie(1993), Andersen(1996) and (Gallant-1996)
- Other methods use Kalman filters in Harvey(1994), Harvey(1996) and Kim(1996)
- There is a method of moments conditioned to a diffusion equation in Bollerslev(2002)
- Bayesian methods like MCMC in Jacquier(1994), Eraker(2001) and Kim(1999)
- MLE (*Maximum Likelihood Approach*) in Aït-Sahalia(2007)
- Methods that use empirical characteristics functions in Singleton(2001)
- Another very wide range variety of methods use empirical calibration through implied volatility surfaces in the market or from the value of a set of tradable option prices , all of that even with heuristic, deterministic optimisation or stochastic optimisation innovations, e.g. in Gathered(2006), Fatone(2014), Hurn(2015).

Objective Function

- The Heston model - under physical measure - requires to calibrate the parameters $\mu, \theta, \sigma, \kappa, \rho$ and v_0.
- This study will primarily focus on: the first five parameters values and consequently it is assumed that $v_0 = \theta$.
- So if $x = (\mu, \theta, \sigma, \kappa, \rho) \in \mathbb{R}^5$, empirical calibration to USD/COP consists of constructing an O.F.: $\Psi(x) = (\Xi \circ \Lambda \circ \Gamma)(x) = \frac{\sum_{i=1}^{m} h_i}{m}$,
- $\Psi(x)$ aimed to obtain the probability of rejecting the null hypothesis H_0 : *the two samples, real returns & each one of the simulated returns for the exchange rate are drawn from the same continuous distribution.*

Objective Function - Implementation

- $\Gamma : \mathbb{R}^5 \longrightarrow \mathbb{R}_{m \times n}$, builds a matrix $m \times n$ with m USDCOP returns simulated trajectories under Heston model, and for n successive days
- $\Lambda : \mathbb{R}_{m \times n} \longrightarrow \mathbb{R}_m$ performs a two-sample KS test: USDCOP returns & each of the m simulated trajectories
  The function $\Lambda(Y)$ returns a KS test $p - value$ for each trajectory, and a decision is taken: accept H_0 and do $h_i = 0$ if $p_i \geq 0.05$, or reject H_0 and do $h_i = 1$, with a significance level of 0.05. Finally,
- $\Xi : \mathbb{R}_m \longrightarrow [0, 1]$ returns the probability of rejecting the null hypothesis H_0

Objective Function - Notes

- Under this line of reasoning, some natural questions arise:
  ▷ Is there an optimal value x_{opt} that minimises the function $\Psi(x)$?
  ▷ If so, how to find it?
- If the value x_{opt} exists, it could be a local minimum of the $\Psi(x)$, near to zero but different from it, given the test level of significance of 0.05, and caused by the stochastic nature of the experiment.
- An auxiliary non-linear-optimisation algorithm can be used to find x_{opt}, and such algorithm should use in its routine only values of $\Psi(x)$ function without the use of any gradient, since there are no guaranteed conditions about differentiability of the function.

Objective Function, two-sample KS test and Nelder Mead simplex direct search

- This is therefore a stochastic optimisation problem that is solved by finding a parametric value x_{opt} such that

$$x_{opt} = argmin_x \Psi(x) \ . \tag{3}$$

- The existence of x_{opt} will be determined if the proposed direct search algorithm, *Nelder-Mead simplex direct search* Nelder(1965), Lagarias(1998), Lewis(2000) can find it! This algorithm is implemented in Matlab Optimisation Toolbox under a function called *fminsearch*.

inicial value of $x : x_0$

- First consideration: $(\mu, \theta, \sigma, \kappa, \rho) \in [243 * \min \phi(t), 243 * \max \phi(t)] \times (0, \max \left(\phi^2(t)\right) * 243] \times (0,1] \times (0, \infty) \times [-1,1]$
- Second consideration: $(\mu, \theta, \sigma, \kappa, \rho) = (243 * \overline{\phi(t)}, 243 * var(\phi(t)), 0.5, 10, -0.1)$ ($\rho < 0$: leverage effect, Aït-Sahalia et.al (2013))
- Third consideration: $(\mu, \theta, \sigma, \kappa, \rho) = (243 * \overline{\phi(t)}, 243 * var(\phi(t)), \sigma, \kappa, \rho)$, where: $\sigma = 0.1 : 0.1 : 1.0, \kappa = 1 : 1 : 15, \rho = -0.20 : 0.1 : 0.20$

Figure 4: $\overline{\Psi(x)}$

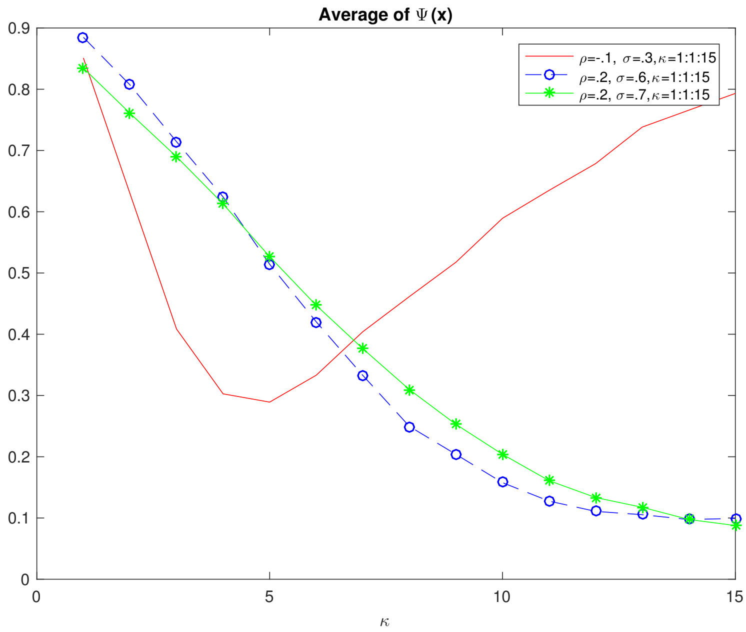


Figure 5: $\overline{\Psi(x)}$

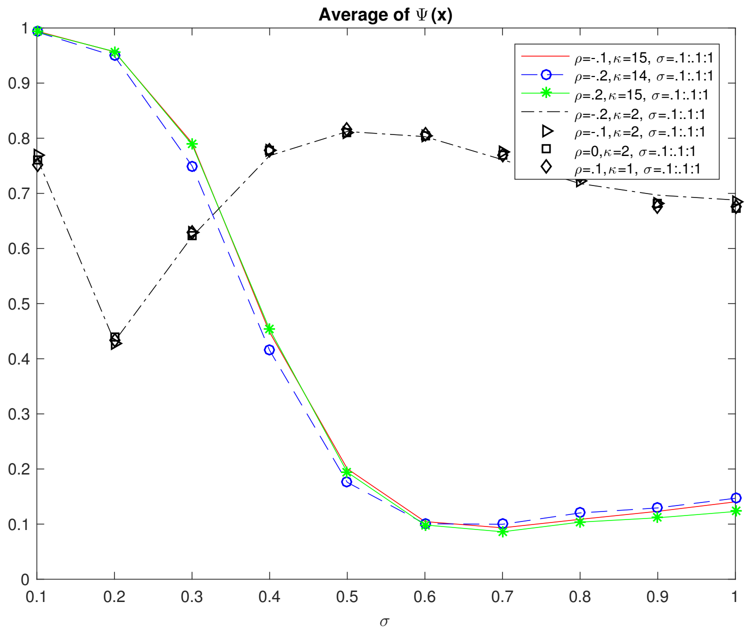


Figure 6: $\overline{\Psi(x)}$

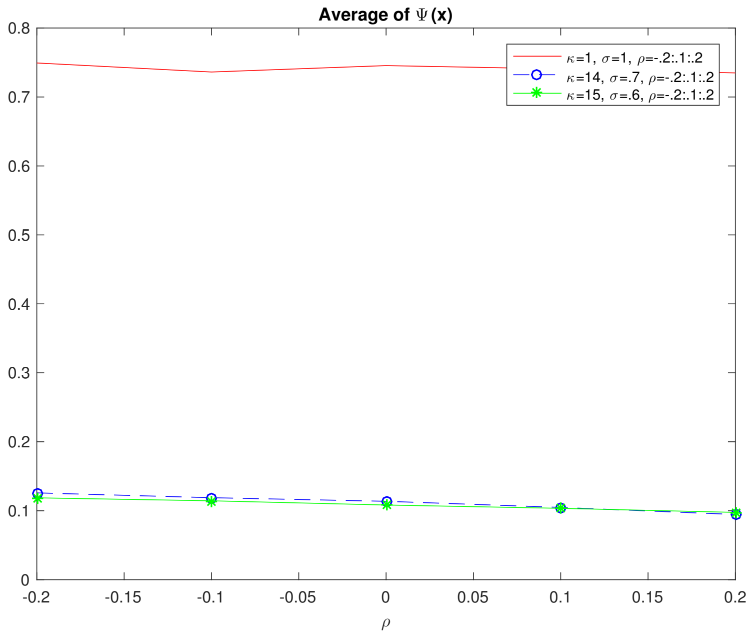


Table 2: Emp. Heston Calibration - Phy. Measure, $(\mu, \theta, \sigma, \kappa, \rho)$

initial value x_0	x_{opt}	$\Psi(x_{opt})$
$(.1268, .0126, .3, 3, -.1)$	$(.1291, .0127, .3129, 2.9489, -.1010)$	$.22 = 22\%$
$(.1268, .0126, .7, 15, .1)$	$(.1271, .0126, .7018, 15.1875, .1028)$	$.02 = 2\%$

Figure 7: $\Psi(x)$

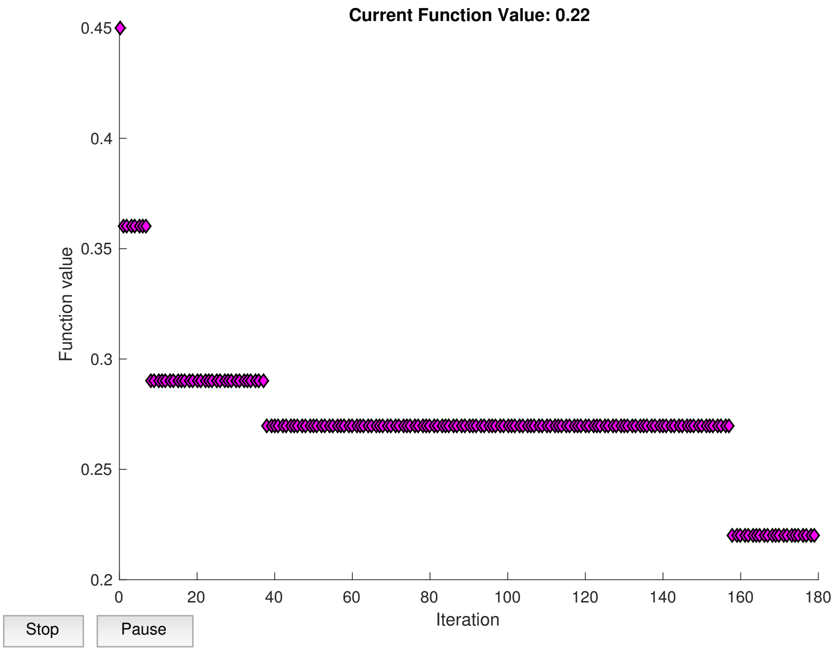


Figure 8: $\Psi(x)$

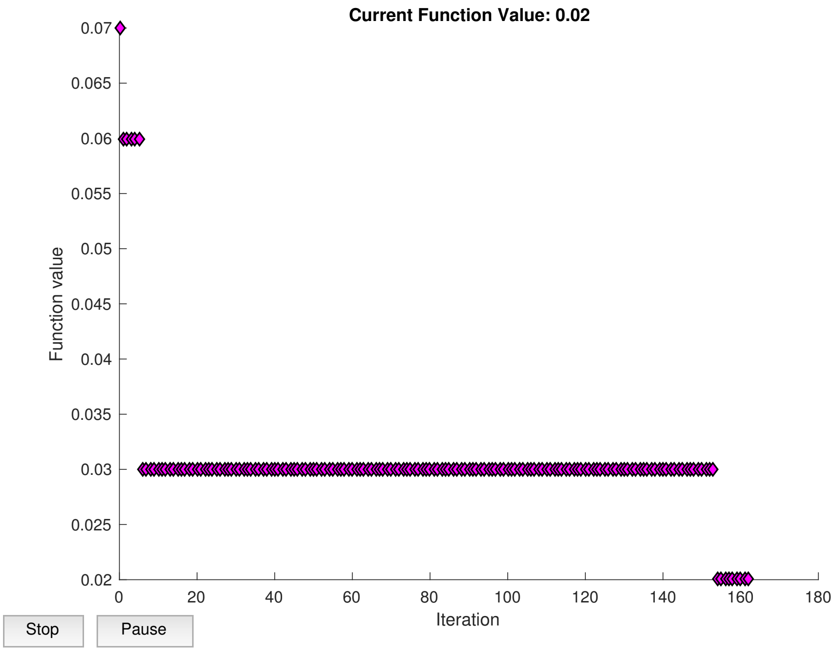
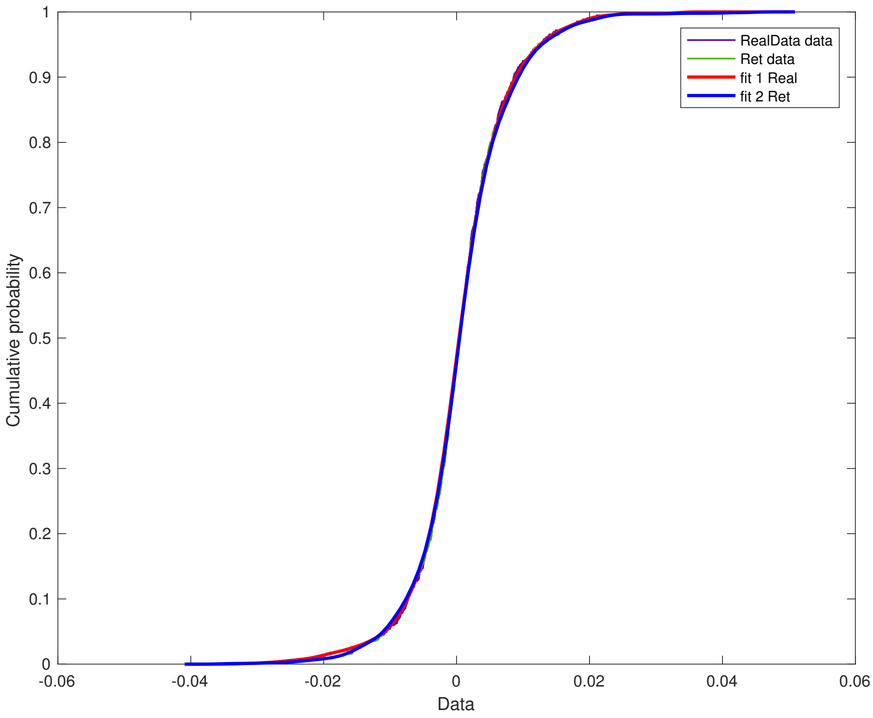
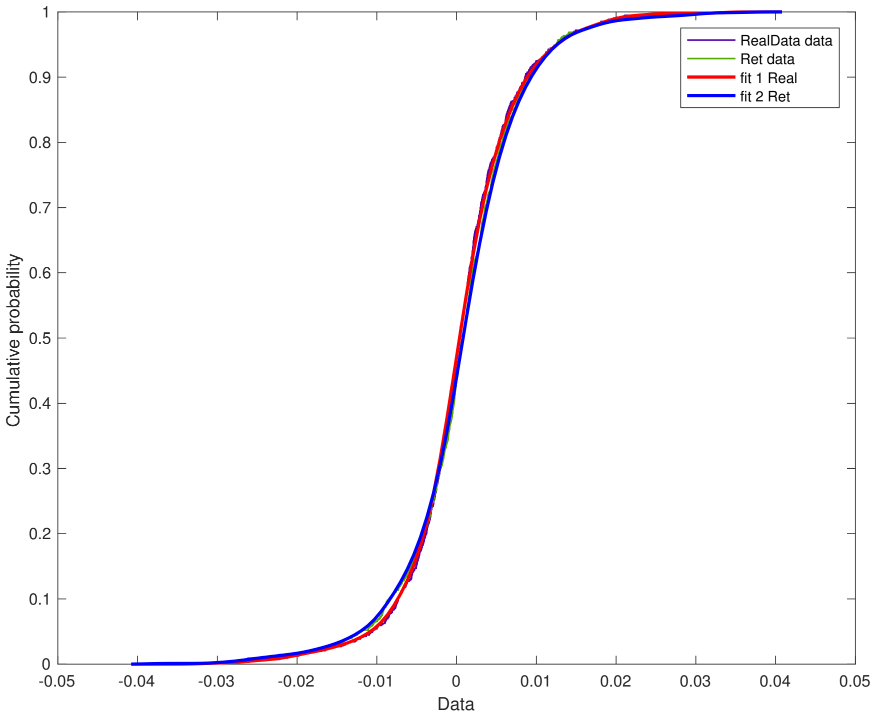


Figure 9: $\Psi(x)$



Results

Figure 10: $\Psi(x)$



Conclusions

- Calibration of Heston SVM (under physical measure of risk) seems to be adequate to describe USDCOP returns
- More than 95% of simulations achieved with FO & Nelder Mead pass two-KS test
- Questions and the future:
  ▷ Comparing with other Heston physical risk measure calibration
  ▷ Leverage effect on calibration of ρ
  ▷ Relationship $\sigma - \kappa$: different combination give similar values of the O.F - Is Ψ flat close to the optimum?
  ▷ Bridge between physical and risk neutral measure - λ?