Almost sure hedging with price impact

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Motivation

- BS and local (stochastic) vol models:
  - Are useful because they provide a clear hedging rule
  - Disregard frictions because do not work at high frequency
  - Taking costs into account would lead to useless degenerate prices/strategies (in theory) and is helpless

However:
- Do not take price impact and illiquidity costs into account
- Problematic when large positions (possibly shared) or illiquid underlying (may run after the delta)

Question: Can we built a model which
- Takes price impact and illiquidity costs into account
- Leads to a clear hedging and pricing rule
- Does not have embedded hidden transaction costs (otherwise the super-hedging price would be degenerate)
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Some references

☐ Many works on hedging with illiquidity or impact: Sircar and Papanicolaou 98, Schönbucher and Wilmot 00, Frey 98, Liu and Yong 05, Cetin, Jarrow and Protter 04, Cetin, Soner and Touzi 09, Almgren and Li 13, Millot and Abergel 11, Guéant and Pu 13,...

☐ Illiquidity + impact + perfect hedging: Loeper 14 (updated in 16, verification arguments).
Impact rule and continuous time trading dynamics
Basic rule (only permanent for the moment): an order of $\delta$ units moves the price by

$$X_{t-} \rightarrow X_t = X_{t-} + \delta f(X_{t-}),$$

and costs

$$\delta X_{t-} + \frac{1}{2} \delta^2 f(X_{t-}) = \delta \frac{X_{t-} + X_t}{2}.$$

We just model the curve around $\delta = 0$ as will pass to continuous time rebalancements (could be more general away from 0).
A trading signal is an Itô process of the form

\[ Y = Y_0 + \int_0^t b_s \, ds + \int_0^t a_s \, dW_s. \]
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Trade at times \( t_i^n = iT/n \) the quantity \( \delta_{t_i^n} = Y_{t_i^n} - Y_{t_{i-1}^n} \).
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The stock price evolves according to

\[ X = X_{t_i^n} + \int_{t_i^n}^t \sigma(X_s) \, dW_s \]

between two trades (can add a drift or be multivariate).
The corresponding dynamics are

\[ Y^n_t := \sum_{i=0}^{n-1} Y^n_{t_i^*} 1\{t_i^* \leq t < t_{i+1}^* \} + Y^n_T 1\{t=T\}, \quad \delta^n_{t_i} = Y^n_{t_i^*} - Y^n_{t_{i-1}^*} \]

\[ X^n = X_0 + \int_0^\cdot \sigma(X^n_s) dW_s + \sum_{i=1}^n 1_{[t_i^*, T]} \delta^n_{t_i} f(X^n_{t_i^-}), \]

\[ V^n = V_0 + \int_0^\cdot Y^n_s dX^n_s + \sum_{i=1}^n 1_{[t_i^*, T]} \frac{1}{2} (\delta^n_{t_i})^2 f(X^n_{t_i^-}), \]

where

\[ V^n = \text{cash part} + Y^n X^n = \text{“portfolio value”}. \]
Passing to the limit $n \to \infty$, it converges in $S_2$ to

\[ Y = Y_0 + \int_0^\cdot b_s ds + \int_0^\cdot a_s dW_s \]

\[ X = X_0 + \int_0^\cdot \sigma(X_s) dW_s + \int_0^\cdot f(X_s) dY_s + \int_0^\cdot a_s(\sigma f')(X_s) ds \]

\[ V = V_0 + \int_0^\cdot Y_s dX_s + \frac{1}{2} \int_0^\cdot a_s^2 f(X_s) ds \]

\[ \left( Y_{t_i}^n - Y_{t_{i-1}}^n \right) f(X_{t_i}^n) \]

\[ \left( Y_{t_i}^n - Y_{t_{i-1}}^n \right)^2 f(X_{t_i}^n) \]

at a speed $\sqrt{n}$. 
Adding a resilience effect

Given a speed of resilience $\rho > 0$,

$$X^n = X_0 + \int_0^\cdot \sigma(X^n_s) dW_s + R^n,$$

$$R^n = R_0 + \sum_{i=1}^n 1_{[t^n_i,T]} \delta^n_{t^n_i} f(X^n_{t^n_i}) - \int_0^\cdot \rho R^n_s ds.$$
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□ The continuous time dynamics becomes

$$X = X_0 + \int_0^\cdot \sigma(X_s)dW_s + \int_0^\cdot f(X_s)dY_s + \int_0^\cdot (a_s(\sigma f')(X_s) - \rho R_s)ds$$

$$R = R_0 + \int_0^\cdot f(X_s)dY_s + \int_0^\cdot (a_s(\sigma f')(X_s) - \rho R_s)ds$$

$$V = V_0 + \int_0^\cdot Y_s dX_s + \frac{1}{2} \int_0^\cdot \dot{a}_s^2 f(X_s)ds.$$. 
The case of covered options

- The premium and payoff are paid in cash and stocks with a number of stocks decided by the trader. Avoids any initial and final market impact.
Resilience does not play any role, we take $\rho \equiv 0$.

\[
dY = adW + bdt
\]
\[
dX = \sigma(X)dW + f(X)dY + a(\sigma f')(X)dt
\]
\[
dV = YdX + \frac{1}{2}a_s^2 f(X)dt.
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    dX &= \sigma(X)dW + f(X)dY + a(\sigma f')(X)dt \\
    dV &= YdX + \frac{1}{2}a_s^2 f(X)dt.
\end{align*}
$$

Super-hedging price:

$$
v(t, x) := \inf \{ v = c + yx : c, \phi = (a, b, y) \text{ s.t. } V_T^{t,x,v,\phi} \geq g(X_T^{t,x,\phi}) \},
$$
Let us assume that we use the delta-hedging rule:

\[ V = v(\cdot, X), \quad Y = \partial_x v(\cdot, X). \]

Then, equating the \( dt \) terms implies

\[
\frac{1}{2} a^2 f(X) = \partial_t v(\cdot, X) + \frac{1}{2} (\sigma + af)^2(X) \partial_{xx}^2 v(\cdot, X),
\]

and applying Itô’s Lemma to \( Y - \partial_x v(\cdot, X) \) leads to

\[
\frac{a}{\sigma + fa} = \partial_{xx}^2 v(\cdot, X).
\]

By a little bit of algebra:

\[
\left[ -\partial_t v - \frac{1}{2} \sigma^2 (1 - f \partial_{xx}^2 v) \partial_{xx}^2 v \right](\cdot, X) = 0.
\]
The pricing pde should be

\[-\partial_t v - \frac{1}{2} \frac{\sigma^2}{(1 - f \partial_{xx} v)} \partial_{xx} v = 0 \quad \text{on} \quad [0, T) \times \mathbb{R},\]

\[v(T-, \cdot) = g \quad \text{on} \quad \mathbb{R}.\]
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**Singular pde**: take care of $1 \neq f \partial_{xx}^2 v$
- Can find smooth solutions s.t. $1 > f \partial_{xx}^2 v$, cf. Loeper 16 (under conditions).
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- Can find smooth solutions s.t. \(1 > f \partial_{xx}^2 v\), cf. Loeper 16 (under conditions).
- One possibility: add a gamma constraint \(\partial_{xx}^2 v \leq \bar{\gamma}\) with \(f\bar{\gamma} < 1\).
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- Can find smooth solutions s.t. \(1 > f \partial_{xx}^2 v\), cf. Loeper 16 (under conditions).
- One possibility : add a gamma constraint \(\partial_{xx}^2 v \leq \tilde{\gamma}\) with \(f \tilde{\gamma} < 1\).
- A constraint of the form \(f \partial_{xx}^2 v > 1\) does not make sense.
Hedging with a gamma constraint

Reformulation of the dynamics

\[ dY = \gamma^a(X) dX + \mu_{\gamma^a}^b(X) dt \quad \text{and} \quad dX = \sigma^a(X) dW + \mu_X^{a,b}(X) dt. \]

\[ \square \] We now define \( \nu \) with respect to the gamma constraint

\[ \gamma^a(X) \leq \tilde{\gamma}(X) \]

with

\[ f\tilde{\gamma} < 1 - \varepsilon, \quad \varepsilon > 0. \]
Pricing pde:

\[
\min \left\{ -\partial_t v - \frac{1}{2} \frac{\sigma^2}{(1-f \partial_{xx} v)} \partial_{xx}^2 v, \tilde{\gamma} - \partial_{xx}^2 v \right\} = 0 \quad \text{on } [0, T) \times \mathbb{R}.
\]

Propagation of the gamma constraint at the boundary:

\[
v(T-, \cdot) = \hat{g} \quad \text{on } \mathbb{R}
\]

with \( \hat{g} \) the smallest (viscosity) super-solution of

\[
\min \left\{ \varphi - g, \tilde{\gamma} - \partial_{xx}^2 \varphi \right\} = 0.
\]
Pricing pde:

$$\min \left\{ -\partial_t v - \frac{1}{2} \frac{\sigma^2}{(1 - f \partial_{xx}^2 v)} \partial_{xx}^2 v , \tilde{\gamma} - \partial_{xx}^2 v \right\} = 0 \quad \text{on } [0, T) \times \mathbb{R}.$$ 

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Perfect hedging: Smooth solution under additional conditions, leading to perfect hedging by following $Y = \partial_x v(\cdot, X)$. 

$\square$
Stochastic target approach

- Geometric DPP (Soner and Touzi): for all stopping time $\theta$ with values in $[0, T]$

\[ V_0 \geq v(0, X_0) \text{ “if and only if” } V_{\theta} \geq v(\theta, X_{\theta}) \text{ for some } (a, b, Y_0). \]
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- “Only if” part ok : $v$ is a super-solution (use weak formulation + small time behavior of double stochastic integrals).
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- Problem. Can not prove the "if" part:
  - at \( \theta \) we have a position \( Y_\theta \) that may not match with the position \( \hat{Y}_\theta \) associated to \( v(\theta, X_\theta) \). Can not jump from \( Y_\theta \) to \( \hat{Y}_\theta \) because of the impact...
  - use the pde smoothing approach of B. and Nutz: produce a smooth supersolution \( v^\varepsilon \) arbitrarily close to the (viscosity) solution \( v^0 \) of the pde and use a verification argument. Then,

\[
\begin{align*}
\text{pde comparison} & \quad \varepsilon \to 0 & \text{super-hedging from } v^\varepsilon \\
\underbrace{v \geq v^0} & \quad \underbrace{v^\varepsilon \geq v}.
\end{align*}
\]
The case of uncovered options

- Has an *initial* impact when build the initial position in stocks and a *final impact* when liquidate it at the end.
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- Has an initial impact when build the initial position in stocks and a final impact when liquidate it at the end.

- Super-hedging price \( w = \) minimal initial cash so that

\[
V_T - Y_T X_T \geq g_0(X_T) \quad \text{and} \quad Y_T = g_1(X_T).
\]

(Recall that \( V = \text{cash} + YX \))
Adding jumps and splitting of large orders

We now consider a trading signal of the form

\[ Y = Y_0 + \int_0^\tau b_s \, ds + \int_0^\tau a_s \, dW_s + \int_0^\tau \delta \nu(d\delta, ds) \]
Adding jumps and splitting of large orders

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\[ Y = Y_0 - \int_0^\tau b_s ds + \int_0^\tau a_s dW_s + \int_0^\tau \delta \nu(d\delta, ds) \]

Jumps \( \delta_i \) at time \( \tau_i \) is passed on \([\tau_i, \tau_i + \varepsilon]\) at a rate \( \delta_i / \varepsilon \).
The limit dynamics when $\varepsilon \to 0$ is

\[
X = X_0 - \int_0^\cdot \sigma(X_s)dW_s + \int_0^\cdot f(X_s)dY_s^c + \int_0^\cdot a_s \sigma f'(X_s)ds \\
+ \int_0^\cdot \int_0^\cdot \Delta x(X_{s-}, \delta) \nu(d\delta, ds)
\]

\[
V = V_0 - \int_0^\cdot Y_s dX_s^c + \frac{1}{2} \int_0^\cdot a_s^2 f(X_s)ds \\
+ \int_0^\cdot \int_0^\cdot (Y_{s-} \Delta x(X_{s-}, \delta) + J(X_{s-}, \delta)) \nu(d\delta, ds).
\]

in which

\[
\Delta x(x, \delta) + x = x(x, \delta) := x + \int_0^\delta f(x(x, s))ds
\]

and $J(x, \delta) := \int_0^\delta sf(x(x, s))ds$. 
Dynamic programming

□ Modified geometric dynamic programming:

\[ v \geq w(0, x) \]

“if and only if”

\[ V_\theta \geq w(\theta, x(X_\theta, -Y_\theta)) + J(x(X_\theta, -Y_\theta), Y_\theta) \text{ for some } (a, b, \nu), \ y \in \mathbb{R} \]
A quasi-linear pde

\[ 0 = -\partial_t w - \hat{\mu}(\cdot, \hat{y}) \partial_x [w + \mathcal{I}] - \frac{1}{2} \hat{\sigma}(\cdot, \hat{y})^2 \partial_{xx}^2 [w + \mathcal{I}] \]

where

\[ \hat{\mu}(\cdot, y) := \frac{1}{2} [\partial_{xx} x \sigma^2](x(\cdot, y), -y) \quad \text{and} \quad \hat{\sigma}(\cdot, y) := (\sigma \partial_x x)(x(\cdot, y), -y), \]

and

\[ \hat{y}(t, x) := x^{-1}(x, x + f(x) \partial_x w(t, x)). \]
Pricing equation

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\[ G(x) := \inf \{ yx(x, y) + g_0(x(x, y)) - \mathcal{I}(x, y) : y = g_1(x(x, y)) \}. \]
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\[ 0 = -\partial_tw - \hat{\mu}(\cdot, \hat{y})\partial_x[w + \mathcal{I}] - \frac{1}{2}\hat{\sigma}(\cdot, \hat{y})^2\partial_{xx}^2[w + \mathcal{I}] \]

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and

\[ \hat{y}(t, x) := x^{-1}(x, x + f(x)\partial_xw(t, x)). \]

□ Terminal condition

\[ G(x) := \inf \{yx(x, y) + g_0(x(x, y)) - \mathcal{I}(x, y) : y = g_1(x(x, y))\}. \]

□ Perfect hedging: Smooth solution under additional conditions, leading to perfect hedging by following \( Y = \hat{y}(\cdot, X). \)
Covered vs Uncovered

Not a simple approximation!

Quasi-linear vs fully non-linear pde

Modified delta vs standard delta hedging rule.

In both cases, perfect hedging is feasible.
Numerical illustration
Constant impact and constraint.

Bachelier model: \( dX_t = 0.2 \, dW_t \).

Butterfly option: \( g(x) = (x + 1)^+ - 2x^+ + (x - 1)^+ \), \( T = 2 \).

Covered option.

Figure: Left: Dashed line: \( f = 0.5, \ \bar{\gamma} = 1.75 \); solid line: \( f = 0, \ \bar{\gamma} = 1.75 \); dotted line: \( f = 0, \ \bar{\gamma} = +\infty \).
Thank you!


