Almost sure hedging with price impact

B. Bouchard

Ceremade - Univ. Paris-Dauphine

Joint works with G. Loeper (Monash Univ.) and Y. Zou (Paris-Dauphine)

Motivation

- BS and local (stochastic) vol models :
  - Are useful because they provide a clear hedging rule
  - Disregard frictions because do not work at high frequency
  - Taking costs into account would lead to useless degenerate prices/strategies (in theory) and is helpless

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- However :
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- Question : Can we built a model which
  - Takes price impact and illiquidity costs into account
  - Leads to a clear hedging and pricing rule
  - Does not have embedded hidden transaction costs (otherwise the super-hedging price would be degenerate)

Some references

□ Many works on hedging with illiquidity or impact : Sircar and Papanicolaou 98, Schönbucher and Wilmot 00, Frey 98, Liu and Yong 05, Cetin, Jarrow and Protter 04, Cetin, Soner and Touzi 09, Almgren and Li 13, Millot and Abergel 11, Guéant and Pu 13,...

□ Illiquidity + impact + perfect hedging : Loeper 14 (updated in 16, verification arguments).

Impact rule and continuous time trading dynamics

Impact rule

□ Basic rule (only permanent for the moment) : an order of δ units moves the price by

$$X_{t-} \longrightarrow X_t = X_{t-} + \delta f(X_{t-}),$$

and costs

$$\delta X_{t-} + \frac{1}{2}\delta^2 f(X_{t-}) = \delta \frac{X_{t-} + X_t}{2}.$$

□ We just model the curve around $\delta = 0$ as will pass to continuous time rebalancements (could be more general away from 0).

Trading signal and discrete trading dynamics

- A trading signal is an Itô process of the form

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- The stock price evolves according to

$$X = X_{t_i^n} + \int_{t_i^n}^{\cdot} \sigma(X_s) dW_s$$

between two trades (can add a drift or be multivariate).

- The corresponding dynamics are

$$Y_t^n := \sum_{i=0}^{n-1} Y_{t_i^n} \mathbf{1}_{\{t_i^n \le t < t_{i+1}^n\}} + Y_T \mathbf{1}_{\{t=T\}} \ , \ \delta_{t_i^n}^n = Y_{t_i^n}^n - Y_{t_{i-1}^n}^n$$

$$X^n = X_0 + \int_0^{\cdot} \sigma(X_s^n) dW_s + \sum_{i=1}^{n} \mathbf{1}_{[t_i^n, T]} \delta_{t_i^n}^n f(X_{t_i^n-}^n),$$

$$V^n = V_0 + \int_0^{\cdot} Y_{s-}^n dX_s^n + \sum_{i=1}^{n} \mathbf{1}_{[t_i^n, T]} \frac{1}{2} (\delta_{t_i^n}^n)^2 f(X_{t_i^n-}^n),$$

where

$$V^n = \text{ cash part } + Y^n X^n = \text{"portfolio value"}.$$

□ Passing to the limit $n \to \infty$, it converges in $\mathbf{S}_2$ to

$$Y = Y_0 + \int_0^{\cdot} b_s ds + \int_0^{\cdot} a_s dW_s$$

$$X = X_0 + \int_0^{\cdot} \sigma(X_s) dW_s + \underbrace{\int_0^{\cdot} f(X_s) dY_s + \int_0^{\cdot} a_s(\sigma f')(X_s) ds}_{(Y_{t_i^n}^n - Y_{t_{i-1}^n}^n) f(X_{t_i^n -}^n)}$$

$$V = V_0 + \int_0^{\cdot} Y_s dX_s + \frac{1}{2} \underbrace{\int_0^{\cdot} a_s^2 f(X_s) ds}_{(Y_{t_i^n}^n - Y_{t_{i-1}^n}^n)^2 f(X_{t_i^n -}^n)},$$

at a speed $\sqrt{n}$.

Adding a resilience effect

- Given a speed of resilience $\rho > 0$,

$$X^n = X_0 + \int_0^{\cdot} \sigma(X_s^n) dW_s + R^n,$$

$$R^n = R_0 + \sum_{i=1}^{n} \mathbf{1}_{[t_i^n, T]} \delta_{t_i^n}^n f(X_{t_i^n -}^n) - \int_0^{\cdot} \rho R_s^n ds.$$

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- The continuous time dynamics becomes

$$X = X_0 + \int_0^{\cdot} \sigma(X_s) dW_s + \int_0^{\cdot} f(X_s) dY_s + \int_0^{\cdot} (a_s(\sigma f')(X_s) - \rho R_s) ds$$

$$R = R_0 + \int_0^{\cdot} f(X_s) dY_s + \int_0^{\cdot} (a_s(\sigma f')(X_s) - \rho R_s) ds$$

$$V = V_0 + \int_0^{\cdot} Y_s dX_s + \frac{1}{2} \int_0^{\cdot} a_s^2 f(X_s) ds.$$

The case of covered options

- The premium and payoff are paid in cash and stocks with a number of stocks decided by the trader. Avoids any initial and final market impact.

Hedging and pricing

- Resilience does not play any role, we take $\rho \equiv 0$.

$$
\begin{aligned}
dY &= a dW + b dt \\
dX &= \sigma(X) dW + f(X) dY + a(\sigma f')(X) dt \\
dV &= Y dX + \frac{1}{2} a_s^2 f(X) dt.
\end{aligned}
$$

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- Super-hedging price :

$$\mathrm{v}(t,x) := \inf\{v = c + yx \ : \ c, \phi = (a,b,y) \text{ s.t. } V_T^{t,x,v,\phi} \geq g(X_T^{t,x,\phi})\},$$

Let us assume that we use the delta-hedging rule :

$$V = \mathrm{v}(\cdot, X) \quad , \quad Y = \partial_x \mathrm{v}(\cdot, X).$$

Then, equating the dt terms implies

$$\frac{1}{2} a^2 f(X) = \partial_t \mathrm{v}(\cdot, X) + \frac{1}{2} (\sigma + af)^2 (X) \partial^2_{xx} \mathrm{v}(\cdot, X),$$

and applying Itô's Lemma to $Y - \partial_x \mathrm{v}(\cdot, X)$ leads to

$$\frac{a}{\sigma + fa} = \partial^2_{xx} \mathrm{v}(\cdot, X).$$

By a little bit of algebra :

$$\left[-\partial_t \mathrm{v} - \frac{1}{2} \frac{\sigma^2}{(1 - f \partial^2_{xx} \mathrm{v})} \partial^2_{xx} \mathrm{v} \right] (\cdot, X) = 0.$$

The pricing pde should be

$$
\begin{aligned}
-\partial_t \mathrm{v} - \frac{1}{2}\frac{\sigma^2}{(1 - f\partial^2_{xx}\mathrm{v})}\partial^2_{xx}\mathrm{v} = 0 \quad \text{on } [0, T) \times \mathbb{R}, \\
\mathrm{v}(T-, \cdot) = g \quad \text{on } \mathbb{R}.
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- One possibility : add a gamma constraint $\partial^2_{xx}\mathrm{v} \leq \bar{\gamma}$ with $f\bar{\gamma} < 1$.
- A constraint of the form $f\partial^2_{xx}\mathrm{v} > 1$ does not make sense.

Hedging with a gamma contraint

Reformulation of the dynamics

$$dY = \gamma^a(X)dX + \mu_Y^{a,b}(X)dt \ \text{ and } \ dX = \sigma^a(X)dW + \mu_X^{a,b}(X)dt.$$

□ We now define v with respect to the gamma constraint

$$\gamma^a(X) \leq \bar{\gamma}(X)$$

with

$$f\bar{\gamma} < 1 - \varepsilon, \ \ \varepsilon > 0.$$

Pricing pde :

$$\min \left\{ -\partial_t \mathrm{v} - \frac{1}{2} \frac{\sigma^2}{(1 - f \partial^2_{xx} \mathrm{v})} \partial^2_{xx} \mathrm{v} \;,\; \bar{\gamma} - \partial^2_{xx} \mathrm{v} \right\} = 0 \quad \text{on } [0, T) \times \mathbb{R}.$$

Propagation of the gamma contraint at the boundary :

$$\mathrm{v}(T-, \cdot) = \hat{g} \quad \text{on } \mathbb{R}$$

with $\hat{g}$ the smallest (viscosity) super-solution of

$$\min \left\{ \varphi - g \;,\; \bar{\gamma} - \partial^2_{xx} \varphi \right\} = 0.$$

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□ Perfect hedging : Smooth solution under additional conditions, leading to perfect hedging by following $Y = \partial_x \mathrm{v}(\cdot, X)$.

Stochastic target approach

☐ Geometric DPP (Soner and Touzi) : for all stopping time θ with values in $[0, T]$

$$V_0 \geq \mathrm{v}(0, X_0) \text{ “if and only if” } V_\theta \geq \mathrm{v}(\theta, X_\theta) \text{ for some } (a, b, Y_0).$$

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- use the pde smoothing approach of B. and Nutz : produce a smooth supersolution v^ε arbitrarily close to the (viscosity) solution v^0 of the pde and use a verification argument. Then,

$$\underbrace{\mathrm{v} \geq v^0}_{\textbf{pde comparison}} \quad \underset{\varepsilon \to 0}{\longleftarrow} \quad \underbrace{v^\varepsilon \geq \mathrm{v}}_{\textbf{super-hedging from } v^\varepsilon} \quad .$$

The case of uncovered options

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□ Super-hedging price w= minimal initial cash so that

$$V_T - Y_T X_T \geq g_0(X_T) \text{ and } Y_T = g_1(X_T).$$

(Recall that $V = \text{cash} + YX$)

Adding jumps and splitting of large orders

- We now consider a trading signal of the form

$$Y = Y_{0-} + \int_0^{\cdot} b_s ds + \int_0^{\cdot} a_s dW_s + \int_0^{\cdot} \delta\nu(d\delta, ds)$$

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- Jumps δ_i at time τ_i is passed on $[\tau_i, \tau_i + \varepsilon]$ at a rate δ_i / ε.

- The limit dynamics when $\varepsilon \to 0$ is

$$\begin{aligned} X &= X_{0-} + \int_0^{\cdot} \sigma(X_s)dW_s + \int_0^{\cdot} f(X_s)dY_s^c + \int_0^{\cdot} a_s \sigma f'(X_s)ds \\ &\quad + \int_0^{\cdot} \int \Delta \mathrm{x}(X_{s-}, \delta)\nu(d\delta, ds) \\ V &= V_{0-} + \int_0^{\cdot} Y_s dX_s^c + \frac{1}{2}\int_0^{\cdot} a_s^2 f(X_s)ds \\ &\quad + \int_0^{\cdot} \int \left(Y_{s-}\Delta \mathrm{x}(X_{s-}, \delta) + \mathfrak{I}(X_{s-}, \delta)\right)\nu(d\delta, ds). \end{aligned}$$

in which

$$\Delta \mathrm{x}(x, \delta) + x = \mathrm{x}(x, \delta) := x + \int_0^{\delta} f(\mathrm{x}(x, s))ds$$

$$\text{and} \quad \mathfrak{I}(x, \delta) := \int_0^{\delta} s f(\mathrm{x}(x, s))ds.$$

Dynamic programming

- Modified geometric dynamic programming :

$$v \geq w(0, x)$$

"if and only if"

$$V_\theta \geq w(\theta, \mathrm{x}(X_\theta, -Y_\theta)) + \mathfrak{I}(\mathrm{x}(X_\theta, -Y_\theta), Y_\theta) \text{ for some } (a, b, \nu),\ y \in \mathbb{R}$$

Pricing equation

- A quasi-linear pde

$$0 = -\partial_t w - \hat{\mu}(\cdot, \hat{y})\partial_x[w + \mathfrak{I}] - \tfrac{1}{2}\hat{\sigma}(\cdot, \hat{y})^2\partial^2_{xx}[w + \mathfrak{I}]$$

where

$$\hat{\mu}(\cdot, y) := \frac{1}{2}[\partial^2_{xx}\mathrm{x}\sigma^2](\mathrm{x}(\cdot, y), -y) \text{ and } \hat{\sigma}(\cdot, y) := (\sigma\partial_x\mathrm{x})(\mathrm{x}(\cdot, y), -y),$$

and

$$\hat{y}(t, x) := \mathrm{x}^{-1}(x, x + f(x)\partial_x w(t, x)).$$

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□ Terminal condition

$$G(x) := \inf\left\{y\mathrm{x}(x, y) + g_0(\mathrm{x}(x, y)) - \mathfrak{I}(x, y) : \; y = g_1(\mathrm{x}(x, y))\right\}.$$

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Covered vs Uncovered

Not a simple approximation !

Quasi-linear vs fully non-linear pde

Modified delta vs standard delta hedging rule.

In both cases, perfect hedging is feasible.

Numerical illustration

- Constant impact and constraint.
- Bachelier model : $dX_t = 0.2\,dW_t$.
- Butterfly option : $g(x) = (x+1)^+ - 2x^+ + (x-1)^+,\ T = 2$.

Covered option.

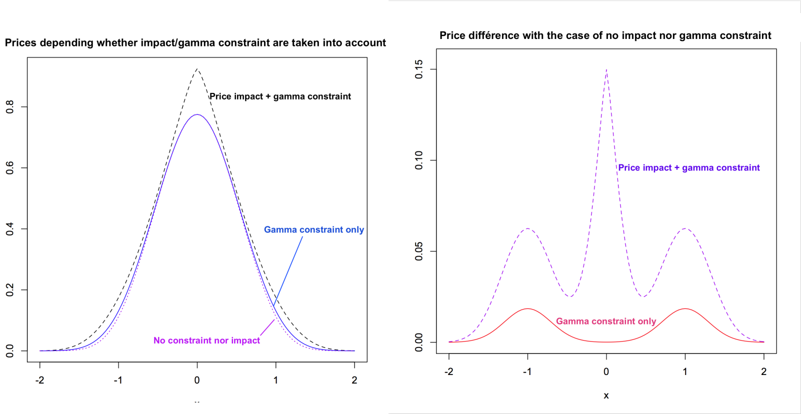

Figure : Left : Dashed line : $f = 0.5,\ \bar{\gamma} = 1.75$; solid line : $f = 0,\ \bar{\gamma} = 1.75$; dotted line : $f = 0,\ \bar{\gamma} = +\infty$.

Thank you !

B. Bouchard, G. Loeper, and Y. Zou.
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