Heston vs Heston

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Implied volatility

- Asset price process: \((S_t = e^{X_t})_{t \geq 0}\), with \(X_0 = 0\).
- No dividend, no interest rate.
- Black-Scholes-Merton (BSM) framework:
  \[
  C_{BS}(\tau, k, \sigma) := \mathbb{E}_0 \left( e^{X_\tau} - e^k \right)_+ = \mathcal{N}(d_+) - e^k \mathcal{N}(d_-),
  \]
  \[
  d_{\pm} := -\frac{k}{\sigma \sqrt{\tau}} \pm \frac{1}{2} \sigma \sqrt{\tau}.
  \]
- Spot implied volatility \(\sigma_\tau(k)\): the unique (non-negative) solution to
  \[
  C_{\text{observed}}(\tau, k) = C_{BS}(\tau, k, \sigma_\tau(k)).
  \]
- Implied volatility: unit-free measure of option prices.
Which model to calibrate the volatility surface?

- Stochastic volatility models (Heston, 3/2, Stein-Stein): easy to simulate (even for path-dependent options), realistic overall implied volatility surface.

- Lévy / jump / affine models: provide a better short-dated fit to the implied volatility surface (steeper skew for out-of-the-money Puts), but jumps notoriously difficult to hedge.

- Local volatility models: not easy to calibrate to the observed surface.

- Local-stochastic volatility models: very appealing, but technically challenging.
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However the implied volatility is not available in closed form for most models. Its asymptotic behaviour (small/large $k, \tau$) however provides us with closed-form approximations, information about the impact of parameters....
Implied volatility ($\sigma_\tau(k)$) asymptotics as $|k| \uparrow \infty$, $\tau \downarrow 0$ or $\tau \uparrow \infty$:

- Mijatović-Tankov (2012): small-$\tau$ for jump models.

**Related works:**

- Kim, Kunitomo, Osajima, Takahashi (1999-...): asymptotic expansions based on Kusuoka-Yoshida-Watanabe method (expansion around a Gaussian).
- Lorig-Pagliarani-Pascucci (2014-···): expansions around the money ($k = 0$).

*Note: from expansions of densities to implied volatility asymptotics is ‘automatic’ (Gao-Lee (2013)).*
Take-away summary

- Classical stochastic volatility models generate a constant short-maturity ATM skew and a large-maturity one proportional to $\tau^{-1}$;

- However, short-term data suggests a time decay of the ATM skew proportional to $\tau^{-\alpha}$, $\alpha \in (0, 1/2)$.

- One solution: adding volatility factors (Gatheral’s Double Mean-Reverting, Bergomi-Guyon), each factor acting on a specific time horizon. But risk of over-parameterisation of the model.

- In the Lévy case (Tankov, 2010), the situation is different, as $\tau \downarrow 0$:
  - in the pure jump case with $\int_{(-1,1)} |x|\nu(dx) < \infty$, then $\sigma_\tau^2(0) \sim c\tau$;
  - in the ($\alpha$) stable case, $\sigma_\tau^2(0) \sim c\tau^{1-2/\alpha}$ for $\alpha \in (1, 2)$;
  - for out-of-the-money options, $\sigma_\tau^2(k) \sim \frac{k^2}{2\tau|\log(\tau)|}$.
  - But short term at-the-money skew: constant.
Rough volatility models

• In 2014, Gatheral-Jaisson-Rosenbaum (+ Bayer-Gatheral-Friz) proposed a fractional volatility model:

\[\begin{align*}
    dS_t &= S_t \sigma_t dZ_t, \\
    d\sigma_t &= \eta dW_t^H,
\end{align*}\]  \hspace{0.5cm} (1)

where \(W^H\) is a fractional Brownian motion.

• Time series of the Oxford-Man SPX realised variance as well as implied volatility smiles of the SPX suggest that \(H \in (0, 1/2)\): short-memory volatility.

• Is not statistically rejected by Ait-Sahalia-Jacod’s test (2009) for Itô diffusions.

• Main drawback: loss of Markovianity \((H \neq 1/2)\) rules out PDE techniques, and Monte Carlo is computationally intensive.

An intuitive remark: Mandelbrot-van Ness representation for fBm:

\[W_t^H = \int_{-\infty}^0 K_1(s, t)dW_s + \int_0^t K_2(s, t)dW_s.\]

At time zero, the volatility process in (1) has already accumulated some randomness.
(Classical) Stochastic volatility models: calibration?

Consider the Heston model for the log-stock price process $X := \log(S)$:

$$
\begin{align*}
    dX_t &= -\frac{1}{2} V_t \, dt + \sqrt{V_t} \, dB_t, \\
    dV_t &= \kappa (\theta - V_t) \, dt + \xi \sqrt{V_t} \, dW_t, \\
    d\langle B, W \rangle_t &= \rho \, dt,
\end{align*}
$$

where $X_0 = 0$, $V_0 = v_0 > 0$.

All parameters can be calibrated by (local/global) minimisation given an observed BS$^{-1}$ $(C_{\text{observed}}(K, T)|F_{t_0})$ discrete implied volatility surface at time $t_0 = 0$.

- $\theta$: long-term average;
- $\kappa$: mean-reversion speed;
- $\xi$: volatility of volatility;
- $\rho$: leverage effect;
- $v_0$: initial variance.
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\begin{align*}
\text{d}X_t &= -\frac{1}{2} V_t \text{d}t + \sqrt{V_t} \text{d}B_t, & X_0 &= 0,
\text{d}V_t &= \kappa (\theta - V_t) \text{d}t + \xi \sqrt{V_t} \text{d}W_t, & V_0 &= v_0 > 0,
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What is $v_0$?

- For the trader: small-maturity (one week) ATM implied volatility;
- Really observable? Or just another parameter?
- What does $t_0$ actually mean? What about yesterday?
- Yesterday is under $\mathbb{P}$. The option price is under $\mathbb{Q}$.  

Initial???????????
The Heston $\mathcal{H}(\nu)$ model

Under the risk-neutral measure $\mathbb{Q}$:

\[
\begin{align*}
    dX_t &= -\frac{1}{2}V_t dt + \sqrt{V_t} dB_t, \\
    dV_t &= \kappa (\theta - V_t) dt + \xi \sqrt{V_t} dW_t, \\
    d\langle B, W \rangle_t &= \rho dt,
\end{align*}
\]

with $\kappa, \theta, \xi > 0$ and $\rho \in (-1, 1)$.

**Assumption**

- $\nu \parallel \mathcal{F}_0$;
- $\nu$ is a continuous random variable supported on $(\nu_-, \nu_+) \subset [0, \infty]$;
- $M_\nu(u) := \mathbb{E}(e^{u\nu})$, for all $u \in D_\nu := \{ u \in \mathbb{R} : \mathbb{E}(e^{u\nu}) < \infty \} \supset (-\infty, 0]$;
- $D_\nu$ contains at least an open neighbourhood of the origin:
  $m := \sup \{ u \in \mathbb{R} : M_\nu(u) < \infty \}$ belongs to $(0, \infty]$.

When $\nu$ is a Dirac distribution ($\nu_- = \nu_+$), the system (2) corresponds to the classical Heston model. Clearly $S = e^X$ is a $\mathbb{Q}$-martingale.

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Heston vs Heston
Asymptotic behaviour: a review

- **Black-Scholes:**

\[ \text{d}X_t = -\frac{\Sigma^2}{2}\text{d}t + \Sigma\text{d}W_t \text{ with } \Lambda^{\text{BS}}_t(u) := \log \mathbb{E}(e^{uX_t}) = \frac{u(u-1)\Sigma^2 t}{2}. \]

\[ \Lambda^{\text{BS}}(u) := \lim_{t \downarrow 0} t\Lambda^{\text{BS}}_t \left( \frac{u}{t} \right) = \frac{u^2\Sigma^2}{2}, \quad \text{for all } u \in \mathcal{D}_{\text{BS}} = \mathbb{R}. \]

In the language of Large deviations, \( X \sim \text{LDP}(t, \Lambda^*_{\text{BS}}): \)

\[ \mathbb{P}(X_t \in B) \sim \exp \left\{ -\frac{1}{t} \inf_{x \in B} \Lambda^*_{\text{BS}}(x) \right\}, \quad \text{where } \Lambda^*_{\text{BS}}(x) := \sup_{u \in \mathcal{D}_{\text{BS}}} \{ux - \Lambda(u)\} = \frac{x^2}{2\Sigma^2}. \]
Asymptotic behaviour: a review

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- **Classical Heston:**

\[
\Lambda^{H(v_0)}(u) := \lim_{t \downarrow 0} t \Lambda^{H(v_0)}_t \left( \frac{u}{t} \right), \quad \text{for all } u \in \mathcal{D}_H \subset \mathbb{R},
\]

and likewise, \( X \sim \text{LDP} \left( t, \Lambda^*_H(v_0) \right) \): \( \mathbb{P}(X_t \in B) \sim \exp \left\{ -\frac{1}{t} \inf_{x \in B} \Lambda^*_H(v_0)(x) \right\} \).
Asymptotic behaviour: a review

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In the language of Large deviations, \(X \sim \text{LDP}(t, \Lambda^{\text{BS}}_t)\):

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and likewise, \(X \sim \text{LDP}(t, \Lambda^{H(v_0)}_t)\): \(\mathbb{P}(X_t \in B) \sim \exp\left\{-\frac{1}{t} \inf_{x \in B} \Lambda^{H(v_0)}_t(x)\right\}\).

- ‘Equating’ both probabilities above yields

\[
\lim_{t \downarrow 0} \sigma^2_t(x) = \frac{x^2}{2\Lambda^{H(v_0)}_t(x)}, \quad \text{for all } x \neq 0.
\]
Heston, simple case: \( \text{supp}(\mathcal{V}) = (v_-, v_+), v_+ < \infty \)

\[
\Lambda_t^H(u) := \mathbb{E}\left( e^{u X_t^H} \right) = \mathbb{E}\left[ \mathbb{E}\left( e^{u X_t^H} \right) | \mathcal{V} \right] = \mathbb{E}\left( e^{\phi(t,u) + \psi(t,u)} \right) = e^{\phi(t,u)} M_V(\psi(t,u)).
\]

so that

\[
\Lambda^H(u) := \lim_{t \downarrow 0} t\Lambda_t^H\left( \frac{u}{t} \right) = \Lambda^{H(v_+)}(u), \quad \text{for all } u \in \mathcal{D}_H,
\]

and therefore

\[
P(X_t \in B) \sim \exp\left\{ -\frac{1}{t} \inf_{x \in B} \Lambda^*_H(v_+)(x) \right\},
\]

and

\[
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Heston, simple case: \( \text{supp}(V) = (v_-, v_+), \ v_+ < \infty \)

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\Lambda^\mathcal{H}_t(u) := \mathbb{E}\left( e^{uX_t^\mathcal{H}} \right) = \mathbb{E}\left[ \mathbb{E}\left( e^{uX_t^\mathcal{H}} \right) | V \right] = \mathbb{E}\left( e^{\phi(t,u)+\psi(t,u)V} \right) = e^{\phi(t,u)}M_V(\psi(t,u)).
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\]

Remarks:

- This could correspond to a trader’s view: she does not exactly observe the small-time ATM implied volatility, but only a proxy, and \( V \) could be uniformly distributed in \( (v_-, v_+) = (v^\text{atm}_0 - \varepsilon, v^\text{atm}_0 + \varepsilon) \).

- Related to J.P. Fouque’s recent result on uncertain \( v_0 \).

- Intuition: only the right tail of the distribution of \( V \) matters (for the asymptotics).
Heston, general case, Part I: $v_+ = \infty$

Theorem (J-Shi (2016))

Let $h(t) \equiv t^\gamma$, with $\gamma \in (0, 1]$. As $t$ tends to zero, the following pointwise limits hold:

$$
\lim_{t \downarrow 0} h(t) \Lambda_t^H \left( \frac{u}{h(t)} \right) = \begin{cases} 
0, & u \in \mathbb{R}, \text{ if } \gamma \in (0, 1/2), \text{ for any } V, \\
0, & u \in \mathbb{R}, \text{ if } \gamma \in [1/2, 1), \text{ for any } V, \\
\Lambda_{H(v_+)}(u), & u \in D_H, \text{ if } \gamma = 1, \text{ for any } V, \\
0, & u \in D_m, \text{ if } \gamma = 1/2, \text{ for any } V, \quad m < \infty,
\end{cases}
$$

and is infinite elsewhere, where $D_m := (-\sqrt{2m}, \sqrt{2m})$. Whenever $\gamma > 1$ (for any $V$), or $m < \infty$ and $\gamma > 1/2$, the pointwise limit is infinite everywhere except at the origin.
Heston, general case, Part II: $v_+ = \infty$, $m = \infty$

**Assumption A:** $v_+ = \infty$ and $\mathcal{N}$ admits a smooth density $f$ with $\log f(v) \sim -l_1 v^{l_2}$ as $v$ tends to infinity, for some $(l_1, l_2) \in \mathbb{R}_+^* \times (1, \infty)$. 
Heston, general case, Part II: $v_+ = \infty$, $m = \infty$

**Assumption A:** $v_+ = \infty$ and $\nu$ admits a smooth density $f$ with $\log f(\nu) \sim -l_1 \nu^{l_2}$ as $\nu$ tends to infinity, for some $(l_1, l_2) \in \mathbb{R}_+^* \times (1, \infty)$.

**Theorem (J-Shi (2016)): Thin-tail case**

The only non-degenerate speed factor is $\gamma_0 = l_2/(1 + l_2) \in (1/2, 1)$, and, for any $u \in \mathbb{R}$, 

$$
\Lambda_{\gamma_0}(u) := \lim_{t \downarrow 0} t^{\gamma_0} \Lambda_t^{H} \left( \frac{u}{t^{\gamma_0}} \right) = \left( 1 - \frac{1}{l_2} \right) \left( \frac{1}{l_1 l_2} \right)^{\frac{1}{l_2-1}} \left( \frac{u^2}{2} \right)^{\frac{l_2}{l_2-1}}.
$$

$$
\Lambda_{\gamma_0}^*(x) := \sup_{u \in \mathbb{R}} \{ u x - \Lambda_{\gamma_0}(u) \} = cx^{2\gamma_0}, \quad \text{with } c := (2l_2)^{-\frac{l_2}{1+l_2}} l_1^{\frac{1}{1+l_2}} (1 + l_2),
$$

**Corollary**

Under **Assumption A**, $X \sim \text{LDP} \left( t^{\gamma_0}, \Lambda_{\gamma_0}^* \right)$ and $\lim_{t \downarrow 0} t^{\frac{1}{1+l_2}} \sigma_t^2(x) = \frac{x^{2(1-\gamma_0)}}{2c}$. 
A slight detour via moderate deviations

**Theorem (J-Shi (2016)): Thin-tail case**

Let $X_t^\alpha := t^{-\alpha} X_t$. Under **Assumption A**, let $\gamma_1 := (l_2 - 1)^{-1} l_2$.

(i) For any $\gamma \in (0, \gamma_1)$, set $\alpha = (1 - \gamma / \gamma_0) / 2$; then $(X_t^\alpha)_{t \geq 0}$ satisfies a LDP with speed $t^{\gamma}$ and good rate function $\Lambda^*_\gamma$;

(ii) if $\gamma = \gamma_1$, set $\alpha = 1 - \gamma_1$, then $(X_t^\alpha)_{t \geq 0} \sim \text{LDP}(t^{\gamma}, L^*)$ with

$$L^*(x) := \sup_{u \in \mathcal{D}_H} \left\{ ux - \left(1 - \frac{1}{l_2}\right) \left(\frac{1}{l_1 l_2}\right)^{\frac{1}{l_2 - 1}} \Lambda_{H(1)}(u) \right\}^\gamma_1 \right\}, \text{ for all } x \in \mathbb{R}.$$

**Remark:** When $\gamma \in (\gamma_0, \gamma_1)$, then $\alpha < 0$ and (i) yields estimates of the form

$$\mathbb{P}(X_t^\alpha \leq x) = \mathbb{P}(X_t \leq xt^\alpha) \sim \exp\left\{- \frac{\Lambda^*_\gamma(x)}{t^{\gamma}}\right\}, \text{ small time and large strike regime},$$

see also Friz-Gerhold-Pinter (2016).
The fat-tail case

**Assumption B:** There exists \((\gamma_0, \gamma_1, \gamma_2, \omega) \in \mathbb{R}_+^* \times \mathbb{R} \times \mathbb{R} \times \mathbb{N}_+^*\), such that the following asymptotics holds for the log-mgf of \(\mathcal{V}\) as \(u\) tends to \(m\) from below:

\[
\log M_{\mathcal{V}}(u) = \begin{cases} 
\gamma_1 \log(m - u) + \gamma_2 + o(1), & \text{for } \omega = 1, \gamma_1 < 0, \\
\gamma_0 \left[1 + \gamma_1(m - u) \log(m - u) + \gamma_2(m - u) + o(m - u)\right] \frac{1}{(m - u)^{\omega-1}}, & \text{for } \omega \geq 2,
\end{cases}
\]

(3)

**Theorem (J-Shi (2016)): Fat-tail case**

Under **Assumption B**, the implied volatility behaves as follows as \(t\) tends to zero:

\[
\sigma_t^2(k) = \begin{cases} 
\frac{|k|}{2\sqrt{2mt}} + o\left(\frac{1}{t^{1/2}}\right), & \text{for } \omega = 1, \\
\frac{|k|}{2\sqrt{2mt}} + \frac{c_1(k)}{4mt^{1/4}} + o\left(\frac{1}{t^{1/4}}\right), & \text{for } \omega = 2.
\end{cases}
\]

(4)

**Example:** Ergodic distribution of the CIR process (Gamma distribution).
Non-central $\chi^2(q, \lambda)$

$$
M_V(u) = \frac{1}{(1 - 2u)^{q/2}} \exp \left( \frac{\lambda u}{1 - 2u} \right), \quad \text{for all } u \in D_V = \left( -\infty, \frac{1}{2} \right),
$$

- $v_+ = +\infty$;
- $m = 1/2$;
- the only suitable scale function is $h(t) \equiv \sqrt{t}$. 

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Introduction
The Heston model
Examples

Non-central $\chi^2(q, \lambda)$

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**Remark** [Link with Forward-start options / smile in Heston]: Let $X \sim H(v_0)$, and fix a forward-starting date $t > 0$.

$$\mathbb{E} \left[ \left( \frac{S_{t+\tau}}{S_t} - e^x \right)_+ \left| \mathcal{F}_0 \right. \right] = \mathbb{E} \left[ \left( e^{X_{t+\tau} - X_t} - e^x \right)_+ \left| \mathcal{F}_0 \right. \right] =: \mathbb{E} \left[ \left( e^{X^{(t)}_{\tau}} - e^x \right)_+ \left| \mathcal{F}_0 \right. \right],$$

where $(X^{(t)}_{\tau})_{\tau \geq 0} \sim \mathcal{H}(V)$, with

$$V \sim \beta_t \chi^2 \left( q = \frac{4\kappa \theta}{\xi^2}, \lambda = \frac{v_0 e^{-\kappa t}}{\beta_t} \right),$$

and $\beta_t := \frac{\xi^2}{4\kappa} (1 - e^{-\kappa t})$. 

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Heston vs Heston
Folded Gaussian; \( \mathcal{V} \sim |\mathcal{N}(0, 1)| \)

The density of \( \mathcal{V} \) reads

\[
f(\nu) = \sqrt{\frac{2}{\pi}} \exp\left(-\frac{1}{2} \nu^2\right), \quad \text{for all } \nu \in D_\mathcal{V} = \mathbb{R}_+,
\]

which satisfies Assumption A. Simple computations yield

\[
M_\mathcal{V}(z) = 2 \exp\left(\frac{z^2}{2}\right) \mathcal{N}(z), \quad \text{for any } z \in \mathbb{R}.
\]

Direct computations show that the only not-trivial scaling is \( h(t) \equiv t^{2/3} \), and

\[
\lim_{t \downarrow 0} t^{1/3} \sigma_t^2(x) = \frac{(2x)^{2/3}}{3}, \quad \text{for all } x \neq 0.
\]
Folded Gaussian; $\nu \sim |\mathcal{N}(0, 1)|$
Uniform distribution

Vol surface comparison: rHeston (Uniform distribution) vs classic Heston
Buena Suerte Para las Semifinales