Heston vs $\mathcal{H}$eston

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Implied volatility

- Asset price process: $(S_t = \mathrm{e}^{X_t})_{t\geq 0}$, with $X_0 = 0$.
- No dividend, no interest rate.
- Black-Scholes-Merton (BSM) framework:

$$C_{\mathrm{BS}}(\tau, k, \sigma) := \mathbb{E}_0 \left(\mathrm{e}^{X_\tau} - \mathrm{e}^k\right)_+ = \mathcal{N}\left(d_+\right) - \mathrm{e}^k \mathcal{N}\left(d_-\right),$$

$$d_\pm := -\frac{k}{\sigma\sqrt{\tau}} \pm \frac{1}{2}\sigma\sqrt{\tau}.$$

- Spot implied volatility $\sigma_\tau(k)$: the unique (non-negative) solution to

$$C_{\mathrm{observed}}(\tau, k) = C_{\mathrm{BS}}(\tau, k, \sigma_\tau(k)).$$

- Implied volatility: unit-free measure of option prices.

Which model to calibrate the volatility surface?

- Stochastic volatility models (Heston, 3/2, Stein-Stein): easy to simulate (even for path-dependent options), realistic overall implied volatility surface.
- Lévy / jump / affine models: provide a better short-dated fit to the implied volatility surface (steeper skew for out-of-the-money Puts), but jumps notoriously difficult to hedge.
- Local volatility models: not easy to calibrate to the observed surface.
- Local-stochastic volatility models: very appealing, but technically challenging.

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However the implied volatility is not available in closed form for most models. Its asymptotic behaviour (small/large k, τ) however provides us with closed-form approximations, information about the impact of parameters....

Implied volatility ($\sigma_\tau(k)$) asymptotics as $|k| \uparrow \infty$, $\tau \downarrow 0$ or $\tau \uparrow \infty$:

- Berestycki-Busca-Florent (2004): small-τ using PDE methods for diffusions.
- Henry-Labordère (2009): small-τ asymptotics using differential geometry.
- Forde et al.(2012), Jacquier et al.(2012): small/ large τ using large deviations.
- Lee (2003), Benaim-Friz (2009), Gulisashvili (2010-2012), Caravenna-Corbetta (2016), De Marco-Jacquier-Hillairet (2013): $|k| \uparrow \infty$.
- Laurence-Gatheral-Hsu-Ouyang-Wang (2012): small-τ in local volatility models.
- Fouque et al.(2000-2011): perturbation techniques for slow and fast mean-reverting stochastic volatility models.
- Mijatović-Tankov (2012): small-τ for jump models.

Related works:

- Kim, Kunitomo, Osajima, Takahashi (1999-...) : asymptotic expansions based on Kusuoka-Yoshida-Watanabe method (expansion around a Gaussian).
- Deuschel-Friz-Jacquier-Violante (CPAM 2014), De Marco-Friz (2014): small-noise expansions using Laplace method on Wiener space (Ben Arous-Bismut approach).
- Lorig-Pagliarani-Pascucci (2014-$\cdots$): expansions around the money ($k = 0$).

Note: from expansions of densities to implied volatility asymptotics is 'automatic' (Gao-Lee (2013)).

Take-away summary

- Classical stochastic volatility models generate a constant short-maturity ATM skew and a large-maturity one proportional to τ^{-1};
- However, short-term data suggests a time decay of the ATM skew proportional to $\tau^{-\alpha}$, $\alpha \in (0, 1/2)$.
- One solution: adding volatility factors (Gatheral's Double Mean-Reverting, Bergomi-Guyon), each factor acting on a specific time horizon. But risk of over-parameterisation of the model.
- In the Lévy case (Tankov, 2010), the situation is different, as $\tau \downarrow 0$:
  - in the pure jump case with $\int_{(-1,1)} |x|\nu(\mathrm{d}x) < \infty$, then $\sigma_\tau^2(0) \sim c\tau$;
  - in the (α) stable case, $\sigma_\tau^2(0) \sim c\tau^{1-2/\alpha}$ for $\alpha \in (1, 2)$;
  - for out-of-the-money options, $\sigma_\tau^2(k) \sim \dfrac{k^2}{2\tau|\log(\tau)|}$.
  - But short term at-the-money skew: constant.

Rough volatility models

- In 2014, Gatheral-Jaisson-Rosenbaum (+ Bayer-Gatheral-Friz) proposed a fractional volatility model:

$$\begin{aligned} \mathrm{d}S_t &= S_t \sigma_t \mathrm{d}Z_t, \\ \mathrm{d}\sigma_t &= \eta \mathrm{d}W_t^H, \end{aligned} \tag{1}$$

  where W^H is a fractional Brownian motion.
- Time series of the Oxford-Man SPX realised variance as well as implied volatility smiles of the SPX suggest that $H \in (0, 1/2)$: **short-memory** volatility.
- Is not statistically rejected by Ait-Sahalia-Jacod's test (2009) for Itô diffusions.
- Main drawback: loss of Markovianity ($H \neq 1/2$) rules out PDE techniques, and Monte Carlo is computationally intensive.

An intuitive remark: Mandelbrot-van Ness representation for fBm:

$$W_t^H = \int_{-\infty}^{0} \mathcal{K}_1(s,t)\mathrm{d}W_s + \int_0^t \mathcal{K}_2(s,t)\mathrm{d}W_s.$$

At time zero, the volatility process in (1) has already accumulated some randomness.

(Classical) Stochastic volatility models: calibration?

Consider the Heston model for the log-stock price process $X := \log(S)$:

$$\begin{aligned} \mathrm{d}X_t &= -\frac{1}{2}V_t\mathrm{d}t + \sqrt{V_t}\mathrm{d}B_t, & X_0 = 0, \\ \mathrm{d}V_t &= \kappa(\theta - V_t)\mathrm{d}t + \xi\sqrt{V_t}\mathrm{d}W_t, & V_0 = v_0 > 0, \\ \mathrm{d}\langle B, W\rangle_t &= \rho\mathrm{d}t, \end{aligned}$$

All parameters can be calibrated by (local/global) minimisation given an observed $\mathrm{BS}^{-1}(C_{\mathrm{observed}}(K,T)|\mathcal{F}_{t_0})$ discrete implied volatility surface at time $t_0 = 0$.

- θ: long-term average;
- κ: mean-reversion speed;
- ξ: volatility of volatility;
- ρ: *leverage effect*;
- v_0: initial variance.

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What is v_0?

- For the trader: small-maturity (one week) ATM implied volatility;
- Really observable? Or just another parameter?
- What does t_0 actually mean? What about yesterday?
- Yesterday is under $\mathbb{P}$. The option price is under $\mathbb{Q}$.

The $\mathcal{H}$eston $\mathcal{H}(\mathcal{V})$ model

Under the risk-neutral measure $\mathbb{Q}$:

$$\begin{aligned} \mathrm{d}X_t &= -\frac{1}{2}V_t\mathrm{d}t + \sqrt{V_t}\mathrm{d}B_t, && X_0 = 0, \\ \mathrm{d}V_t &= \kappa(\theta - V_t)\mathrm{d}t + \xi\sqrt{V_t}\mathrm{d}W_t, && V_0 \sim \mathcal{V}, \\ \mathrm{d}\langle B, W\rangle_t &= \rho\mathrm{d}t, \end{aligned} \tag{2}$$

with $\kappa, \theta, \xi > 0$ and $\rho \in (-1, 1)$.

Assumption

- $\mathcal{V} \perp\!\!\!\perp \mathcal{F}_0$;
- $\mathcal{V}$ is a continuous random variable supported on $(\mathfrak{v}_-, \mathfrak{v}_+) \subset [0, \infty]$;
- $\mathrm{M}_{\mathcal{V}}(u) := \mathbb{E}\left(\mathrm{e}^{u\mathcal{V}}\right)$, for all $u \in \mathcal{D}_{\mathcal{V}} := \{u \in \mathbb{R} : \mathbb{E}\left(\mathrm{e}^{u\mathcal{V}}\right) < \infty\} \supset (-\infty, 0]$;
- $\mathcal{D}_{\mathcal{V}}$ contains at least an open neighbourhood of the origin: $\mathfrak{m} := \sup\{u \in \mathbb{R} : \mathrm{M}_{\mathcal{V}}(u) < \infty\}$ belongs to $(0, \infty]$.

When $\mathcal{V}$ is a Dirac distribution ($\mathfrak{v}_- = \mathfrak{v}_+$), the system (2) corresponds to the classical Heston model. Clearly $S = \mathrm{e}^X$ is a $\mathbb{Q}$-martingale.

Asymptotic behaviour: a review

- **Black-Scholes:** $\mathrm{d}X_t = -\frac{\Sigma^2}{2}\mathrm{d}t + \Sigma \mathrm{d}W_t$ with $\Lambda_t^{\mathrm{BS}}(u) := \log \mathbb{E}\left(\mathrm{e}^{uX_t}\right) = \frac{u(u-1)\Sigma^2 t}{2}$.

$$\Lambda^{\mathrm{BS}}(u) := \lim_{t \downarrow 0} t \Lambda_t^{\mathrm{BS}}\left(\frac{u}{t}\right) = \frac{u^2\Sigma^2}{2}, \qquad \text{for all } u \in \mathcal{D}_{\mathrm{BS}} = \mathbb{R}.$$

In the language of Large deviations, $X \sim \mathrm{LDP}(t, \Lambda_{\mathrm{BS}}^*)$:

$$\mathbb{P}(X_t \in B) \sim \exp\left\{-\frac{1}{t}\inf_{x\in B}\Lambda_{\mathrm{BS}}^*(x)\right\}, \quad \text{where } \Lambda_{\mathrm{BS}}^*(x) := \sup_{u\in\mathcal{D}_{\mathrm{BS}}}\{ux - \Lambda(u)\} = \frac{x^2}{2\Sigma^2}.$$

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- **Classical Heston:**

$$\Lambda^{\mathrm{H}(v_0)}(u) := \lim_{t \downarrow 0} t \Lambda_t^{\mathrm{H}(v_0)}\left(\frac{u}{t}\right), \qquad \text{for all } u \in \mathcal{D}_{\mathrm{H}} \subset \mathbb{R},$$

and likewise, $X \sim \mathrm{LDP}\left(t, \Lambda^*_{\mathrm{H}(v_0)}\right)$: $\mathbb{P}(X_t \in B) \sim \exp\left\{-\frac{1}{t}\inf_{x\in B}\Lambda^*_{\mathrm{H}(v_0)}(x)\right\}$.

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- 'Equating' both probabilities above yields

$$\lim_{t\downarrow 0}\sigma_t^2(x) = \frac{x^2}{2\Lambda_{\mathrm{H}(v_0)}^*(x)}, \qquad \text{for all } x \neq 0.$$

$\mathcal{H}$eston, simple case: $\mathrm{supp}(\mathcal{V}) = (\mathfrak{v}_-, \mathfrak{v}_+)$, $\mathfrak{v}_+ < \infty$

$$\Lambda_t^{\mathcal{H}}(u) := \mathbb{E}\left(\mathrm{e}^{uX_t^{\mathcal{H}}}\right) = \mathbb{E}\left[\mathbb{E}\left(\mathrm{e}^{uX_t^{\mathcal{H}}}\right)|\mathcal{V}\right] = \mathbb{E}\left(\mathrm{e}^{\phi(t,u)+\psi(t,u)\mathcal{V}}\right) = \mathrm{e}^{\phi(t,u)}\mathrm{M}_{\mathcal{V}}(\psi(t,u)).$$

so that

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Remarks:

- This could correspond to a trader's view: she does not **exactly** observe the small-time ATM implied volatility, but only a proxy, and $\mathcal{V}$ could be uniformly distributed in $(\mathfrak{v}_-, \mathfrak{v}_+) = (v_0^{\mathrm{atm}} - \varepsilon, v_0^{\mathrm{atm}} + \varepsilon)$.
- Related to J.P. Fouque's recent result on *uncertain* v_0.
- Intuition: only the right tail of the distribution of $\mathcal{V}$ matters (for the asymptotics).

$\mathcal{H}$eston, general case, Part I: $\mathfrak{v}_+ = \infty$

Theorem (J-Shi (2016))

Let $h(t) \equiv t^\gamma$, with $\gamma \in (0,1]$. As t tends to zero, the following pointwise limits hold:

$$\lim_{t \downarrow 0} h(t) \Lambda_t^{\mathcal{H}} \left(\frac{u}{h(t)} \right) = \begin{cases} 0, & u \in \mathbb{R}, & \text{if } \gamma \in (0, 1/2), & \text{for any } \mathcal{V}, \\ 0, & u \in \mathbb{R}, & \text{if } \gamma \in [1/2, 1), & \mathfrak{v}_+ < \infty, \\ \Lambda_{\mathrm{H}(\mathfrak{v}_+)}(u), & u \in \mathcal{D}_{\mathrm{H}}, & \text{if } \gamma = 1, & \mathfrak{v}_+ < \infty, \\ 0, & u \in \mathcal{D}_{\mathfrak{m}}, & \text{if } \gamma = 1/2, & \mathfrak{v}_+ = \infty, \mathfrak{m} < \infty, \end{cases}$$

and is infinite elsewhere, where $\mathcal{D}_{\mathfrak{m}} := (-\sqrt{2\mathfrak{m}}, \sqrt{2\mathfrak{m}})$. Whenever $\gamma > 1$ (for any $\mathcal{V}$), or $\mathfrak{m} < \infty$ and $\gamma > 1/2$, the pointwise limit is infinite everywhere except at the origin.

$\mathcal{H}$eston, general case, Part II: $\mathfrak{v}_+ = \infty$, $\mathfrak{m} = \infty$

Assumption A: $\mathfrak{v}_+ = \infty$ and $\mathcal{V}$ admits a smooth density f with $\log f(v) \sim -l_1 v^{l_2}$ as v tends to infinity, for some $(l_1, l_2) \in \mathbb{R}_+^* \times (1, \infty)$.

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Theorem (J-Shi (2016)): Thin-tail case

The only non-degenerate speed factor is $\gamma_0 = l_2/(1+l_2) \in (1/2, 1)$, and, for any $u \in \mathbb{R}$,

$$\Lambda_{\gamma_0}(u) := \lim_{t\downarrow 0} t^{\gamma_0}\Lambda_t^{\mathcal{H}}\left(\frac{u}{t^{\gamma_0}}\right) = \left(1 - \frac{1}{l_2}\right)\left(\frac{1}{l_1 l_2}\right)^{\frac{1}{l_2-1}}\left(\frac{u^2}{2}\right)^{\frac{l_2}{l_2-1}}.$$

$$\Lambda_{\gamma_0}^*(x) := \sup_{u\in\mathbb{R}}\{ux - \Lambda_{\gamma_0}(u)\} = cx^{2\gamma_0}, \qquad \text{with } c := (2l_2)^{\frac{-l_2}{1+l_2}}\, l_1^{\frac{1}{1+l_2}}(1+l_2),$$

Corollary

Under **Assumption A**, $X \sim \mathrm{LDP}\left(t^{\gamma_0}, \Lambda_{\gamma_0}^*\right)$ and $\lim_{t\downarrow 0} t^{\frac{1}{1+l_2}}\sigma_t^2(x) = \dfrac{x^{2(1-\gamma_0)}}{2c}$.

A slight detour via moderate deviations

Theorem (J-Shi (2016)): Thin-tail case

Let $X_t^\alpha := t^{-\alpha} X_t$. Under **Assumption A**, let $\gamma_1 := (l_2 - 1)^{-1} l_2$.

(i) For any $\gamma \in (0, \gamma_1)$, set $\alpha = (1 - \gamma/\gamma_0)/2$; then $(X_t^\alpha)_{t \geq 0}$ satisfies a LDP with speed t^γ and good rate function Λ_γ^*;

(ii) if $\gamma = \gamma_1$, set $\alpha = 1 - \gamma_1$, then $(X_t^\alpha)_{t \geq 0} \sim \mathrm{LDP}(t^\gamma, \mathrm{L}^*)$ with

$$\mathrm{L}^*(x) := \sup_{u \in \mathcal{D}_\mathrm{H}} \left\{ ux - \left(1 - \frac{1}{l_2}\right) \left(\frac{1}{l_1 l_2}\right)^{\frac{1}{l_2 - 1}} \Lambda_{\mathrm{H}(1)}(u)^{\gamma_1} \right\}, \quad \text{for all } x \in \mathbb{R}.$$

Remark: When $\gamma \in (\gamma_0, \gamma_1)$, then $\alpha < 0$ and (i) yields estimates of the form

$$\mathbb{P}(X_t^\alpha \leq x) = \mathbb{P}(X_t \leq x t^\alpha) \sim \exp\left\{ -\frac{\Lambda_\gamma^*(x)}{t^\gamma} \right\} \qquad \textit{small time and large strike regime},$$

see also Friz-Gerhold-Pinter (2016).

The fat-tail case

Assumption B: There exists $(\gamma_0, \gamma_1, \gamma_2, \omega) \in \mathbb{R}_+^* \times \mathbb{R} \times \mathbb{R} \times \mathbb{N}_+^*$, such that the following asymptotics holds for the log-mgf of $\mathcal{V}$ as u tends to $\mathfrak{m}$ from below:

$$\log \mathrm{M}_{\mathcal{V}}(u) = \begin{cases} \gamma_1 \log(\mathfrak{m} - u) + \gamma_2 + o(1), & \omega = 1, \gamma_1 < 0, \\ \dfrac{\gamma_0 \left[1 + \gamma_1(\mathfrak{m} - u)\log(\mathfrak{m} - u) + \gamma_2(\mathfrak{m} - u) + o\left(\mathfrak{m} - u\right)\right]}{(\mathfrak{m} - u)^{\omega - 1}}, & \omega \geq 2, \end{cases} \tag{3}$$

Theorem (J-Shi (2016)): Fat-tail case

Under **Assumption B**, the implied volatility behaves as follows as t tends to zero:

$$\sigma_t^2(k) = \begin{cases} \dfrac{|k|}{2\sqrt{2\mathfrak{m}t}} + o\left(\dfrac{1}{t^{1/2}}\right), & \text{for } \omega = 1, \\ \dfrac{|k|}{2\sqrt{2\mathfrak{m}t}} + \dfrac{c_1(k)}{4\mathfrak{m}t^{1/4}} + o\left(\dfrac{1}{t^{1/4}}\right), & \text{for } \omega = 2. \end{cases} \tag{4}$$

Example: Ergodic distribution of the CIR process (Gamma distribution).

Non-central $\chi^2(q, \lambda)$

$$\mathrm{M}_{\mathcal{V}}(u) = \frac{1}{(1-2u)^{q/2}} \exp\left(\frac{\lambda u}{1-2u}\right), \qquad \text{for all } u \in \mathcal{D}_{\mathcal{V}} = \left(-\infty, \frac{1}{2}\right),$$

- $\mathfrak{v}_+ = +\infty$;
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- the only suitable scale function is $h(t) \equiv \sqrt{t}$.

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Remark [Link with Forward-start options / smile in Heston]: Let $X \sim \mathrm{H}(v_0)$, and fix a forward-starting date $t > 0$.

$$\mathbb{E}\left[\left(\frac{S_{t+\tau}}{S_t} - \mathrm{e}^x\right)_+ | \mathcal{F}_0\right] = \mathbb{E}\left[\left(\mathrm{e}^{X_{t+\tau} - X_t} - \mathrm{e}^x\right)_+ | \mathcal{F}_0\right] =: \mathbb{E}\left[\left(\mathrm{e}^{X_\tau^{(t)}} - \mathrm{e}^x\right)_+ | \mathcal{F}_0\right],$$

where $(X_\tau^{(t)})_{\tau \geq 0} \sim \mathcal{H}(\mathcal{V})$, with

$$\mathcal{V} \sim \beta_t \chi^2\left(q = \frac{4\kappa\theta}{\xi^2}, \lambda = \frac{v_0 \mathrm{e}^{-\kappa_t}}{\beta_t}\right),$$

and $\beta_t := \frac{\xi^2}{4\kappa}(1 - \mathrm{e}^{-\kappa t})$.

Folded Gaussian; $\mathcal{V} \sim |\mathcal{N}(0,1)|$

The density of $\mathcal{V}$ reads

$$f(v) = \sqrt{\frac{2}{\pi}} \exp\left(-\frac{1}{2}v^2\right), \qquad \text{for all } v \in \mathcal{D}_{\mathcal{V}} = \mathbb{R}_+,$$

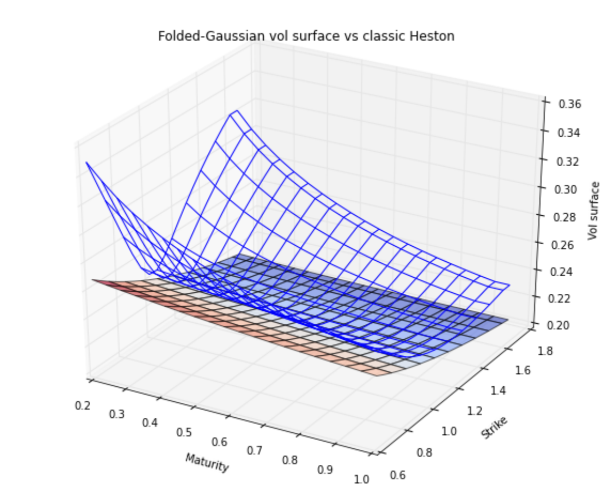
which satisfies **Assumption A**. Simple computations yield

$$\mathrm{M}_{\mathcal{V}}(z) = 2\exp\left(\frac{z^2}{2}\right)\mathcal{N}(z), \qquad \text{for any } z \in \mathbb{R}.$$

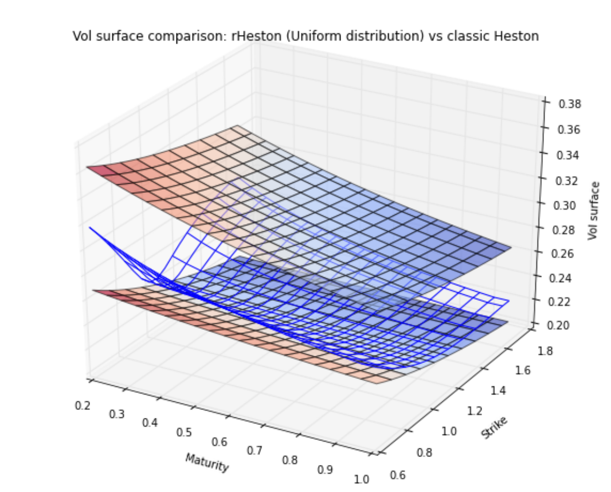
Direct computations show that the only not-trivial scaling is $h(t) \equiv t^{2/3}$, and

$$\lim_{t \downarrow 0} t^{1/3}\sigma_t^2(x) = \frac{(2x)^{2/3}}{3}, \qquad \text{for all } x \neq 0.$$

Folded Gaussian; $\mathcal{V} \sim |\mathcal{N}(0,1)|$



Uniform distribution



Buena Suerte Para las Semifinales