Arbitrage-Free XVA

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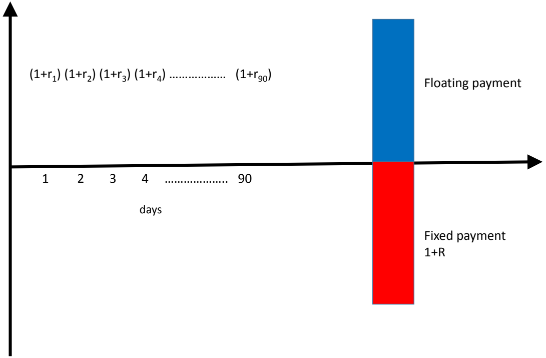
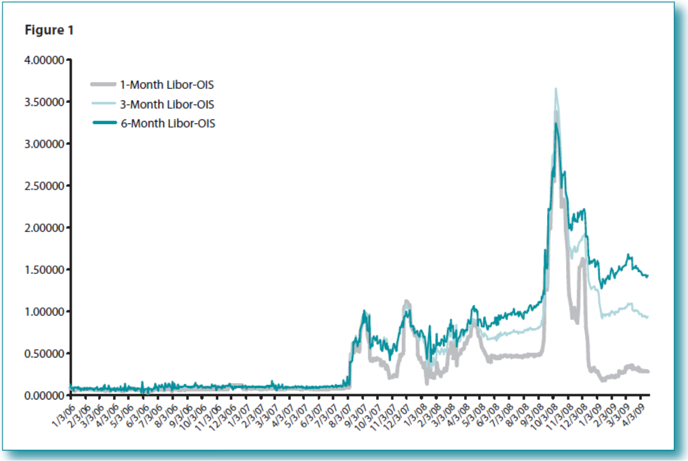
The Overnight Index Swap Rate (OIS) R



The LIBOR-OIS Spread



The LIBOR-OIS Spread

- Consequences
  - Widening of spreads is due to counterparty credit risk
  - LIBOR cannot be considered a risk-free rate any longer
  - One cannot assume the existence of a universal risk-free rate r
  - Rates at which derivatives traders borrow and lend unsecured cash differ
  - How to price and hedge derivatives in presence of funding spreads and counterparty risk?

XVA: Motivation and Challenges

- Dealers need to account for the total costs of their trades
    - funding costs: financing the portfolio of liquid securities used to replicate the traded position
    - collateral costs: funding and retrieving the collateral needed to secure the deal with the counterparty
    - closeout costs: losses incurred if a premature liquidation needs to be executed because the counterparty defaults
- Swap quotes offered to clients reflect the effect of these costs, referred to as XVA
- Many banks (Barclays, JPM, BoA,...) have introduced XVA desks

Related Literature I

- Piterbarg (2010): closed form solution for prices of European style claims, under symmetric rates and default-free counterparties
- Albanese and Andersen (2014): FVA/FDA accounting and its implications for capital requirements, capital valuation adjustments (KVA), etc...
- Bielecki and Rutkowski (2014): no arbitrage pricing in a general semi-martingale framework in presence of funding costs, but without counterparty risk
- Brigo et al. (2012-2015): pricing equations for counterparty risk and funding costs assuming a linear pricing rule

Related Literature II

- Burgard & Kjaer (2010, 2011), Mercurio (2014): PDE representations for XVA
- Crepey (2013): BSDE approach for counterparty risk valuation under funding constraints
- Duffie (2016): FVA is not an adjustment to the swap value, but rather to the value of the bank's equity. Which are the economics implications, incentive effects, etc...?

Overview of Contributions

- Develop a framework to characterize the total valuation adjustment (XVA) of a European style claim
- Derive backward stochastic differential equations (BSDEs) tracking the replicating portfolios of long and short positions in the claim
- Show existence and uniqueness of the BSDE solution, and of a classical solution to the corresponding semi-linear PDE
- Define buyer's and seller's XVA in terms of the unique BSDE solutions
- Develop explicit representations of XVA and of the corresponding hedging strategies

The Market Model

Cash transactions and tradeable securities:

- **Treasury desk**: borrowing and lending at rates r_f^-, r_f^+, respectively
- **Stock** (S_t): used to the hedge market risk of the transaction. Trading happens through repo market at rates r_r^-, r_r^+
- **Risky bonds** (P_t^I, P_t^C): underwritten by investor/counterparty and used to hedge default risk. Bonds are not purchased/sold via the repo market

Stock Short-Selling

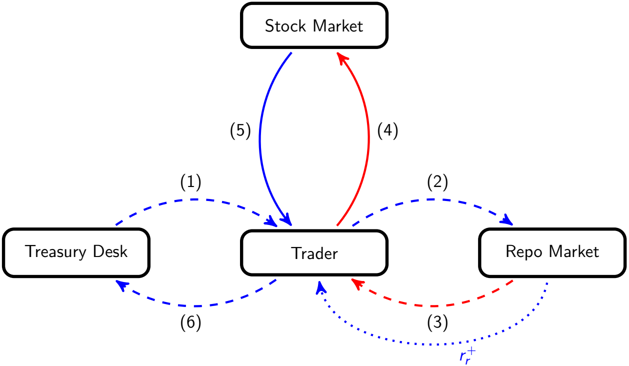


FIGURE: Security driven repo activity: Solid lines are purchases/sales, dashed lines borrowing/lending, dotted lines interest due; blue lines are cash, red lines are stock.

Stock Purchasing

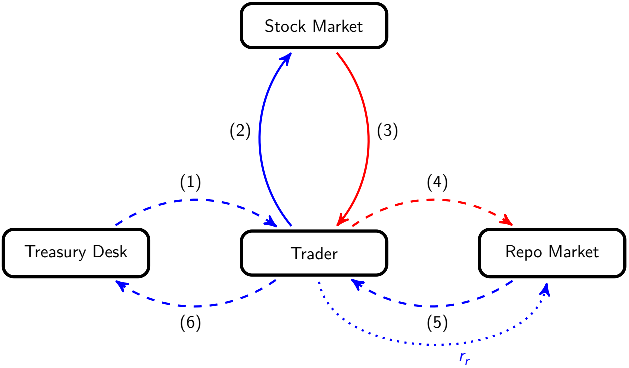


FIGURE: Cash driven repo activity: Solid lines are purchases/sales, dashed lines borrowing/lending, dotted lines interest due; blue lines are cash, red lines are stock.

Price Dynamics

- Under the physical probability measure $\mathbb{P}$, the price processes of the market securities follow

$$
\begin{aligned}
dS_t &= \mu S_t \, dt + \sigma S_t \, dW_t^{\mathbb{P}} \\
dP_t^I &= \mu_I P_t^I \, dt - P_{t-}^I \, d\mathbb{1}_{\{\tau_I \leqslant t\}} \\
&= (\mu_I - h_I^{\mathbb{P}}) P_t^I \, dt - P_{t-}^I \, d\varpi_t^{I,\mathbb{P}} \\
dP_t^C &= \mu_C P_t^C \, dt - P_{t-}^C \, d\mathbb{1}_{\{\tau_C \leqslant t\}} \\
&= (\mu_C - h_C^{\mathbb{P}}) P_t^C \, dt - P_{t-}^C \, d\varpi_t^{C,\mathbb{P}}
\end{aligned}
$$

  for default times τ_I, τ_C with constant intensities $h_I^{\mathbb{P}}$, $h_C^{\mathbb{P}}$ and pure-jump martingales $\varpi^{I,\mathbb{P}}$, $\varpi^{C,\mathbb{P}}$
- $\mathbb{F} := (\mathcal{F}_t)_{t \geqslant 0} = \sigma \left(W_u^{\mathbb{P}}; \; u \leqslant t \right)_{t \geqslant 0}$
- $\mathbb{G} := (\mathcal{G}_t)_{t \geqslant 0} = (\mathcal{F}_t)_{t \geqslant 0} \vee \sigma \left(\mathbb{1}_{\tau_I \leqslant u}, \mathbb{1}_{\tau_C \leqslant u}; \; u \leqslant t \right)_{t \geqslant 0}$

Arbitrage-Free Valuation

- Can we guarantee that there are no arbitrage opportunities in the market?
- As we only model from the point of view of the trader, we can only conclude this from her perspective. . .

Proposition

No-arbitrage conditions:

`Necessary`: $r_r^+ \leqslant r_f^-$, $r_f^+ \leqslant r_f^-$, $r_f^+ < \mu_I$, $r_f^+ < \mu_C$.

`Sufficient`: *Necessary plus* $r_r^+ \leqslant r_f^+ \leqslant r_r^-$

Collateralization

- Collateral is used to secure the derivatives deal
  - Collateral is provided in form of cash (80%)
  - Collateral can be reinvested (rehypothecated) (96%)
  - The collateral provider receives interests at rate r_c^+. The collateral taker pays interests at rate r_c^-.

The Market Model

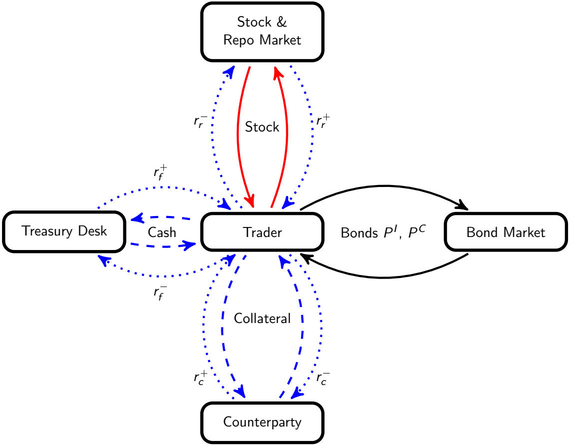


Figure: Solid lines are purchases/sales, dashed lines borrowing/lending, dotted lines interest due; blue lines are cash, red lines stock purchases for cash and black lines bond purchases for cash.

Closeout Payments and Valuation

- The closeout value of the claim is decided by a valuation agent (either party or third party) in accordance with market practices (ISDA)
- The valuation agent determines collateral requirements and closeout value by calculating the "Black-Scholes" price of the transaction
- Such a valuation yields an interest rate r_D
- We can then introduce a valuation measure $\mathbb{Q}$ under which r_D-discounted prices are $\mathbb{Q}$ martingales.
- The XVA will be computed under $\mathbb{Q}$

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- We can then introduce a valuation measure $\mathbb{Q}$ under which r_D-discounted prices are $\mathbb{Q}$ martingales.
- The XVA will be computed under $\mathbb{Q}$
- Remark : A reduced form approach by Andersen, Duffie, Song (2016) postulates the existence of a linear pricing rule and of the corresponding deflator

Collateral and Close-Out Valuation

- Collateral is a percentage α of the price of the contract

$$
\begin{aligned}
C_t &= \alpha \mathbb{1}_{\{\tau_I \wedge \tau_C > t\}} \mathbb{E}^{\mathbb{Q}}\left[e^{-r_D(T-t)} \Phi(S_T) \,\middle|\, \mathcal{F}_t \right] \\
&:= \alpha \mathbb{1}_{\{\tau_I \wedge \tau_C > t\}} \hat{V}(t, S_t)
\end{aligned}
$$

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- Set $\tau = \tau_I \wedge \tau_C \wedge T$. The close-out payment is

$$\begin{aligned} \theta_\tau(\hat{V}) &= \theta_\tau(C, \hat{V}) \\ &:= \hat{V}(\tau, S_\tau) + \mathbb{1}_{\{\tau_C < \tau_I\}} L_C Y^- - \mathbb{1}_{\{\tau_I < \tau_C\}} L_I Y^+, \end{aligned}$$

where $Y := \hat{V}_\tau - C_\tau$ is the residual value of the claim at default

Wealth Process

The dynamics of the wealth process is given by

$$
\begin{aligned}
dV_t = \Big(& r_f^+ \left(\xi_t^f B_t^{r_f}\right)^+ - r_f^- \left(\xi_t^f B_t^{r_f}\right)^- + (r_D - r_r^-)\left(\xi_t S_t\right)^+ \\
& - (r_D - r_r^+)\left(\xi_t S_t\right)^- + r_D \xi_t^I P_t^I + r_D \xi_t^C P_t^C \Big)\, dt \\
& - r_c^- \left(\psi_t^c B_t^{r_c}\right)^+ dt + r_c^+ \left(\psi_t^c B_t^{r_c}\right)^- dt \\
& + \underbrace{(\cdots)}_{\text{martingales}}
\end{aligned}
$$

with $B_t^{r_f}$ funding account, $B_t^{r_c}$ collateral account, ξ_t, and ψ_t number of shares in the securities and various accounts

Arbitrage Pricing

Definition

A price $P \in \mathbb{R}$, of a derivative security with terminal payoff $\xi \in \sigma(S_t; t \leqslant T)$ is called *trader's arbitrage-free*, if for all $\gamma \in \mathbb{R}$ buying γ securities for the price γP and hedging in the market with an admissible strategy does not create trader's arbitrage.

Replicating Wealth

- $V_t^+(\gamma)$: wealth process when replicating the claim $\gamma\Phi(S_T)$, $\gamma > 0$. This means hedging the position after selling γ securities with terminal payoff $\Phi(S_T)$.
- $\left(-V_t^-(\gamma)\right)$: wealth process when replicating the claim $-\gamma\Phi(S_T)$, $\gamma > 0$. This means hedging the position after buying γ securities with terminal payoff $\Phi(S_T)$.

Replicating Wealth Equation

XVA Pricing

A. Capponi

The BSDEs

$$\begin{cases} -dV_t^+(\gamma) &= f^+\big(t, V_t^+, Z_t^+, Z_t^{I,+}, Z_t^{C,+}; \hat{V}\big)\, dt \\ & \quad - Z_t^+\, dW_t^{\mathbb{Q}} - Z_t^{I,+}\, d\varpi_t^{I,\mathbb{Q}} - Z_t^{C,+}\, d\varpi_t^{C,\mathbb{Q}} \\ V_\tau^+(\gamma) &= \gamma\Big(\theta_\tau(\hat{V})\mathbb{1}_{\{\tau<T\}} + \Phi(S_T)\mathbb{1}_{\{\tau=T\}}\Big) \end{cases}$$

$$\begin{cases} -dV_t^-(\gamma) &= f^-\big(t, V_t^-, Z_t^-, Z_t^{I,-}, Z_t^{C,-}; \hat{V}\big)\, dt \\ & \quad - Z_t^-\, dW_t^{\mathbb{Q}} - Z_t^{I,-}\, d\varpi_t^{I,\mathbb{Q}} - Z_t^{C,-}\, d\varpi_t^{C,\mathbb{Q}} \\ V_\tau^-(\gamma) &= \gamma\Big(\theta_\tau(\hat{V})\mathbb{1}_{\{\tau<T\}} + \Phi(S_T)\mathbb{1}_{\{\tau=T\}}\Big) \end{cases}$$

describe the wealth dynamics for buying/selling γ options

Projection Result

Theorem (Crepey and Song (2015))

The $\mathbb{F}$ BSDEs

$$
\begin{aligned}
-dU_t^{sell/buy} &= g^{\pm}(t, U_t^{sell/buy}, \check{Z}_t^{\pm}; \hat{V})\, dt - \check{Z}_t^{\pm}\, dW_t^{\mathbb{Q}} \\
U_T^{sell/buy} &= 0,
\end{aligned}
$$

$$
g^{+}(t, y, \check{z}; \hat{V}) = h_I^{\mathbb{Q}}(\hat{\theta}_I(\hat{V}_t) - y) + h_C^{\mathbb{Q}}(\hat{\theta}_C(\hat{V}_t) - y) + \tilde{f}^{+}(t, y, \check{z}, \hat{\theta}_I(\hat{V}_t) - y, \hat{\theta}_C(\hat{V}_t) - y; \hat{V})
$$

$$
g^{-}(t, y, \check{z}, ; \hat{V}) = -g^{+}(t, -y, -\check{z}; -\hat{V}),
$$

admit a unique solution $(U^{sell/buy}, \check{Z}^{\pm})$, *related to* $(XVA^{sell/buy}, Z^{\pm}, Z^{I,\pm}, Z^{C,\pm})$ *as*

$$
\begin{aligned}
XVA_t^{sell/buy}\mathbb{1}_{\{t<\tau\}} &:= U_t^{sell/buy}\mathbb{1}_{\{t<\tau\}}, \\
Z_t^{\pm} &:= \check{Z}_t^{\pm}\mathbb{1}_{\{t<\tau\}}, \\
Z_t^{I,\pm} &:= \left(\hat{\theta}_I(\hat{V}_t) - U_t^{sell/buy}\right)\mathbb{1}_{\{t<\tau\}}, \\
Z_t^{C,\pm} &:= \left(\hat{\theta}_C(\hat{V}_t) - U_t^{sell/buy}\right)\mathbb{1}_{\{t<\tau\}}.
\end{aligned}
$$

No arbitrage

Theorem

Let Φ be a function of polynomial growth. If $V_0^- \leqslant V_0^+$, then all derivative prices in the closed interval $[\pi^{inf} = V_0^-, V_0^+ = \pi^{sup}]$ are free of trader's arbitrage.

Definition of XVA

Definition

The seller's XVA is defined by

$$\mathrm{XVA}_t^{sell} = V_t^+ - \hat{V}(t, S_t)$$

and the buyer's XVA by

$$\mathrm{XVA}_t^{buy} = V_t^- - \hat{V}(t, S_t).$$

The Extended Piterbarg Model

Extension of Piterbarg's model

- Allow for default of investor and counterparty
- Default risk is hedged by risky bonds
- Maintain Piterbarg's assumption of symmetric rates: $r_f = r_f^+ = r_f^-$, $r_r = r_r^+ = r_r^- = r_D$, $r_c = r_c^+ = r_c^-$
- The total costs of replicating long and short positions coincide, and $\mathrm{XVA}_t^{sell} = \mathrm{XVA}_t^{buy}$

The Extended Piterbarg Model

Proposition (XVA decomposition)

Define $\eta = h_I^{\mathbb{Q}} + h_C^{\mathbb{Q}} + 2r_r - r_f$. *On* $\{\tau > t\}$,

$$
\begin{aligned}
\frac{XVA_t}{\hat{V}_t} &= \underbrace{\Big((r_r - r_f) + \alpha(r_f - r_c)\Big)\frac{1 - e^{-(\eta - r_r)(T-t)}}{\eta - r_r}}_{\text{replicating strategy and collateral costs}} \\
&\quad + \underbrace{\left(r_r - r_f + h_C^{\mathbb{Q}}\right) L_C \frac{1 - e^{-(\eta - r_r)(T-t)}}{\eta - r_r}\Big((1-\alpha)\mathbf{1}_{\hat{V}_t < 0}\Big)}_{\text{costs related to CVA}} \\
&\quad - \underbrace{\left(r_r - r_f + h_I^{\mathbb{Q}}\right) L_I \frac{1 - e^{-(\eta - r_r)(T-t)}}{\eta - r_r}\Big((1-\alpha)\mathbf{1}_{\hat{V}_t > 0}\Big)}_{\text{costs related to DVA}} \\
&:= Adj_t
\end{aligned}
$$

Hedging Strategies

$$
\begin{aligned}
\xi_t &= \mathrm{Adj}_t \hat{\Delta}_t, \\
\xi_t^I &= \frac{\mathrm{XVA}_t + L_I(1-\alpha)(\hat{V}_t)^+}{e^{-(r_D + h_I^{\mathbb{Q}})(T-t)}}, \\
\xi_t^C &= \frac{\mathrm{XVA}_t - L_C(1-\alpha)(\hat{V}_t)^-}{e^{-(r_D + h_C^{\mathbb{Q}})(T-t)}}.
\end{aligned}
$$

The Extended Piterbarg Model

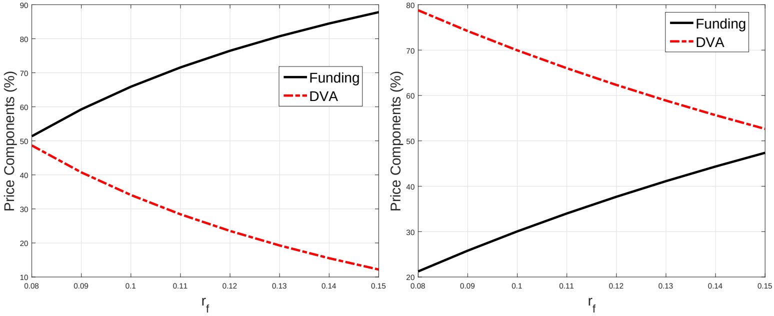
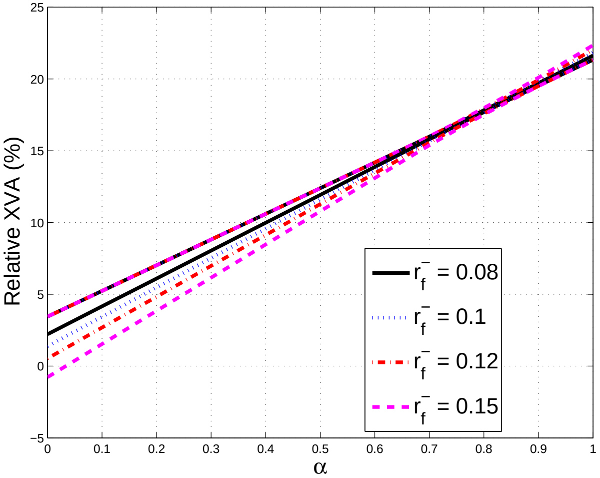


Figure: Left graph: $h_I^{\mathbb{Q}} = 0.15$, $h_C^{\mathbb{Q}} = 0.2$. Right graph: $h_I^{\mathbb{Q}} = 0.5$, $h_C^{\mathbb{Q}} = 0.5$.

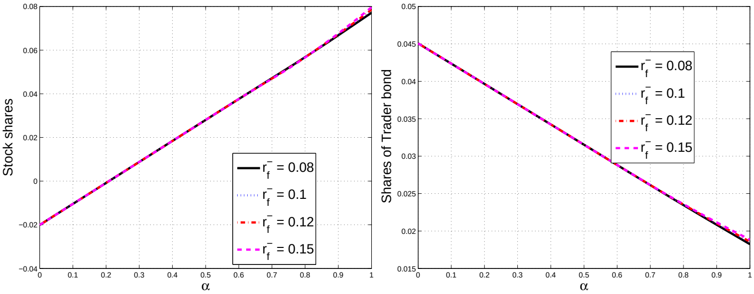
The Impact of Differential Rates

- What if borrowing and lending rates differ?: $r_f^- \neq r_f^+$, $r_r^- \neq r_r^+$, $r_c^- \neq r_c^+$
- BSDE becomes nonlinear: $V_t^+ \neq V_t^-$. We have a no-arbitrage interval for prices
- But, we can use the semilinear PDE representation v associated with the BSDE V for numerical analysis

Band and funding spreads



Hedging strategies



Conclusion

- Developed an arbitrage-free valuation framework for XVA
- Seller's and buyer's XVA characterized as solutions of nonlinear BSDEs
- Funding component of XVA is predominant, but DVA/CVA terms become material if trader/counterparty are very risky
- The no-arbitrage interval widens as funding spreads and collateral levels increase

References

- M. Bichuch, A. Capponi, and S. Sturm. Arbitrage-free XVA. Available at http://papers.ssrn.com/sol3/papers.cfm?abstract_id=2554600

BSDE Relations

- The two BSDEs are intrinsically related:

$$(V_t^-, Z_t^-, Z_t^{I,-}, Z_t^{C,-})$$

is a solution to the data

$$\left(f^-, \theta_\tau(\hat{V}), \Phi(S_T)\right)$$

iff

$$(-V_t^-, -Z_t^-, -Z_t^{I,-}, -Z_t^{C,-})$$

is a solution to the data

$$\left(f^+, \theta_\tau(-\hat{V}), -\Phi(S_T)\right)$$